Mass on a Spring

Let's assume that the motion of a mass on a spring is described by the equation

$$x = A \sin \omega t \tag{1}$$

in which A is the amplitude of the motion, $\omega = 2\pi f$ is the angular frequency (in radians per second) and $x = 0$ when the mass is at rest. Let's calculate the velocity of the mass as a function of time. Recall that velocity is just distance/time. To measure this at an instant we take a small time Δt and record the small distance Δx that the mass travels in this time. Therefore, $v = \Delta x/\Delta t$ and

$$x + \Delta x = A \sin(\omega t + \omega \Delta t). \tag{2}$$

We must now use the trigonometric identities

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \tag{3}$$

and

$$\cos(a + b) = \cos a \cos b - \sin a \sin b \tag{4}$$

Therefore,

$$x + \Delta x = A \sin \omega t \cos \omega \Delta t + A \cos \omega t \sin \omega \Delta t = A \sin \omega t + A \cos \omega t(\omega \Delta t) \tag{5}$$

and $\Delta x = (\omega A \cos \omega t)\Delta t$. We have used the fact that $\sin w \to w$ as $w \to 0$ and $\cos w \to 1$ as $w \to 0$ and that Δt is tiny.

$$v = \omega A \cos \omega t \tag{6}$$

You can do the next step yourself. Acceleration $a = \Delta v/\Delta t$ and

$$a = -\omega^2 A \sin \omega t = -\omega^2 x \tag{7}$$

If the force on the mass is linear, then $F = -kx$. Newton's Law tells us that $F = ma$ (m is the mass); therefore, $F = ma = -\omega^2 mx$. From this equation we can read off the spring constant $k = \omega^2 m$. This means that $\omega = \sqrt{k/m}$. But $\omega = 2\pi f$, and the period $T = 1/f$. Therefore,

$$T = 2\pi \sqrt{\frac{m}{k}} \tag{8}$$

This shows how a mathematical theory predicts that the period of oscillation of a mass on a spring varies with $\sqrt{m}$. All we have assumed is a linear force and Newton's law of acceleration.

Non-linear forces arise for large amplitudes in the pendulum and spring. For brass instruments that occurs when one overblows. In audio we call this distortion.