Kondo effect and non-Fermi liquid behavior in Dirac and Weyl semimetals

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We study the Kondo effect in three-dimensional (3D) Dirac materials and Weyl semimetals. We find the scaling of the Kondo temperature with respect to the doping n and the coupling J between the moment of the magnetic impurity and the carriers of the semimetal. We find that when the temperature is much smaller than the Kondo temperature the resistivity due to the Kondo effect scales as the $n^{-4/3}$. We also study the effect of the interplay of long-range scalar disorder and Kondo effect. In the presence of disorder-induced long-range carrier density inhomogeneities the Kondo effect is not characterized by a Kondo temperature but by a distribution of Kondo temperatures. We obtain the expression of such distribution and show that its features cause the appearance of strong non-Fermi liquid behavior. Finally we compare the properties of the Kondo effect in 3D Dirac materials and 2D Dirac systems like graphene and topological insulators.

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Three-dimensional (3D) Dirac and Weyl semimetals (SMs) [1–4] have been recently realized experimentally [5, 6], thus motivating a great deal of interest on these materials. In Weyl and Dirac SMs [2–4, 7] the conduction and valence bands touch at isolated points of the Brillouin zone (BZ) named "Weyl nodes" (WNs) in Weyl SMs and "Dirac points" (DPs) in Dirac SMs. Around these points the electronic excitations behave as 3D massless Dirac fermions (MDFs) characterized, in the isotropic case, by a density-independent Fermi velocity $v_{\rm F}$. In the case of Weyl SMs the eigenstates of the bare Hamiltonian are non-degenerate [1–4]. Weyl SMs are expected to exhibit unique properties [8–10] and to have surface states forming "Fermi arcs" [2, 3, 11–18]. Conversely, in Dirac SMs the eigenstates are doubly degenerate, *i.e.* each Dirac point corresponds to two copies of overlapping Weyl nodes with opposite chiralities [19] and is protected by the symmetries of the crystal structure. The linear dispersion around the nodes is expected to give rise to anomalous transport properties in both 3D Dirac and Weyl SMs [3, 20]. Graphene [21–23] and the surface states of 3D topological insulators (TIs) [24, 25] constitute the two-dimensional (2D) counterpart of 3D Dirac SMs [24, 25]. While the surface states of TIs are not spin degenerate, and are topologically protected by the large separation in real space, the Dirac cones of graphene are doubly degenerate and are protected by the inversion symmetry of the crystal.

The interaction of dilute magnetic impurities with an electron liquid is one of the most important and studied examples of strongly-correlated physics [26]. The so-called "Kondo effect" [27] is characterized by a temperature scale $T_{\rm K}$. When the temperature (T) is larger than $T_{\rm K}$ the electrons of the host material are only weakly scattered by the impurity. For $T < T_{\rm K}$ the coupling grows non-perturbatively and leads to the formation of a many-body singlet with the electron liquid, which completely screens the impurity magnetic moment.

In this work we show that the unique band structure of 3D Dirac and Weyl SMs strongly affects the nature of the Kondo effect in these systems. We obtain (i) the dependence of $T_{\rm K}$ on the doping level of the SM and on the strength of the antiferromagnetic electron-impurity coupling J, (ii) the correction to the resistivity due to the presence of magnetic impurities in the Kondo regime ($T \to 0$), and find that the interplay of linear dispersion around the nodes, Kondo effect, and long-range scalar disorder induces a non-Fermi liquid (NFL) behavior [28–31] in these systems which can be directly probed by measuring the magnetic response of these materials. Finally we present a systematic comparison of the properties of the Kondo effect between 3D and 2D Dirac SMs [32–45].

In Dirac and Weyl SMs the low-energy states around one of the DPs are described by the Hamiltonian $H_0 = \hbar v_F \hat{c}^\dagger_{\boldsymbol{k}\sigma}(\boldsymbol{k}\cdot\boldsymbol{\tau}_{\sigma\sigma'} - \mu)\hat{c}_{\boldsymbol{k}\sigma'}$ where v_F is the Fermi velocity at the DP, $\hat{c}^\dagger_{\boldsymbol{k}\sigma}$ ($\hat{c}_{\boldsymbol{k}\sigma}$) creates (annihilates) an electron with momentum $\boldsymbol{k}$ and spin (or pseudospin) σ, and μ is the chemical potential. Hereafter we set $\hbar = 1$. For TIs and Weyl SMs $\boldsymbol{\tau}_{\sigma\sigma'}$ is the vector formed by the 2×2 Pauli matrices in spin space. For graphene and 3D Dirac SMs $\boldsymbol{\tau}_{\sigma\sigma'}$ is the vector formed by the 2×2 Pauli matrices in pseudospin space. It is easy to see that the contribution of Fermi arcs to the Kondo effect in Weyl SMs is negligible. Electrons on the Fermi arcs have the spin locked to the momentum. Spin-flip processes can occur only if electrons are scattered to another surface, but these processes are extremely rare. Thus the differences between Weyl and Dirac SMs, besides the extra spin degeneracy $g_s = 2$ of Dirac eigenstates, turn out to be inessential for our purposes.

In the presence of diluted (identical) magnetic impurities, coupled antiferromagnetically to the carriers, the

system is described by the Hamiltonian $H = H_0 + H_J$ where $H_J = J\sum_{\boldsymbol{r},\boldsymbol{R}} \hat{c}^\dagger_{\boldsymbol{r}\sigma}\boldsymbol{\tau}_{\sigma\sigma'}\hat{c}_{\boldsymbol{r}\sigma'}\cdot\boldsymbol{S}\delta(\boldsymbol{r}-\boldsymbol{R})$, with $\boldsymbol{S}$ the magnetic moments of impurities and $\{\boldsymbol{R}\}$ their positions. Here $\hat{c}_{\boldsymbol{r}\sigma}$ ($\hat{c}^\dagger_{\boldsymbol{r}\sigma}$) is the Fourier transform of the operator $\hat{c}_{\boldsymbol{k}\sigma}$ ($\hat{c}^\dagger_{\boldsymbol{k}\sigma}$) in the real-space domain. Since impurities interact only with the electrons of the SM, hereafter we focus on a single magnetic impurity.

To treat the coupling of the magnetic impurity to the free carriers we use a large-N expansion [46, 47] in which $\boldsymbol{S}$ is expressed in terms of auxiliary creation (annihilation) fermionic operators $\hat{f}^\dagger_\sigma$ ($\hat{f}_\sigma$) satisfying the constraint $n_f = \sum_\sigma \hat{f}^\dagger_\sigma\hat{f}_\sigma = 1$, with $\sigma = 1,\ldots,N_\sigma$. We set $N_\sigma = 2$ in the end of the calculation, which corresponds to the case of a magnetic impurity with $|\boldsymbol{S}| = 1/2$. In terms of the $\hat{f}$-operators the coupling term H_J takes the form $H_J = J\sum_{\boldsymbol{k},\boldsymbol{k}',\sigma}\hat{c}^\dagger_{\boldsymbol{k}\sigma}\hat{c}_{\boldsymbol{k}'\sigma'}\hat{f}^\dagger_{\sigma'}\hat{f}_\sigma$.

The large-N expansion allows a mean field treatment of the Kondo problem [46], and is known to return accurate and reliable results for the case of diluted magnetic impurities [46–48]. We decouple the quartic interaction term H_J via a Hubbard-Stratonovich field $s \sim \sum_{\boldsymbol{k},\sigma}\langle\hat{f}^\dagger_\sigma\hat{c}_{\boldsymbol{k}\sigma}\rangle$, which thus describes the hybridization between "localized" ($\hat{f}$) and "itinerant" ($\hat{c}$) electronic states. The constrain $n_f = 1$ is enforced with the introduction of the Lagrange multiplier μ_f, which plays the role of the chemical potential of the f-electrons [see also Eq. (5)]. The resulting action is quadratic in the fermionic fields, and the functional integration over $\hat{f}$ and $\hat{c}$ can be carried out analytically. Approximating s and μ_f as static (mean-)fields, we finally obtain the effective action

$$\mathcal{S}_{\rm eff}=\frac{2}{\pi k_{\rm B}T}\int_{-D-\mu}^{D-\mu} d\varepsilon\; n_{\rm F}(\varepsilon)\arctan\left[\frac{\pi}{2}\frac{|s|^2\mathcal{N}(\varepsilon+\mu)}{\varepsilon-\mu_f}\right] + \frac{1}{k_{\rm B}T}\left(\frac{|s|^2}{J}-\mu_f\right)\;, \tag{1}$$

where $n_{\rm F}(\varepsilon) = (e^{\varepsilon/(k_BT)}+1)^{-1}$ is the Fermi-Dirac occupation factor and $\mathcal{N}(\varepsilon) = VN_{\rm w}\varepsilon^2/(2\pi^2\hbar^3v_{\rm F}^3)$ is the 3D density-of-states (DOS) of electrons in the SM. Here $N_{\rm w}$ the number of DPs, V is the volume of the system and D is a cut-off corresponding to half the bandwidth of the SM. The corresponding effective action for the 2D case is obtained by replacing $\mathcal{N}(\varepsilon) \to VN_{\rm w}|\varepsilon|/(2\pi\hbar^2v_{\rm F}^2)$. By minimizing $S_{\rm eff}$ within the saddle point approximation [47] we obtain the self-consistent equations for $|s|^2$ and μ_f.

$$\begin{aligned}\int_{-D-\mu}^{D-\mu} d\varepsilon\;\frac{n_{\rm F}(\varepsilon)(\varepsilon-\mu_f)\mathcal{N}(\varepsilon+\mu)}{(\varepsilon-\mu_f)^2+(\pi|s|^2\mathcal{N}(\varepsilon+\mu)/2)^2} &= -\frac{1}{J}\;,\\ \int_{-D-\mu}^{D-\mu} d\varepsilon\;\frac{n_{\rm F}(\varepsilon)|s|^2\mathcal{N}(\varepsilon+\mu)}{(\varepsilon-\mu_f)^2+(\pi|s|^2\mathcal{N}(\varepsilon+\mu)/2)^2} &= 1\;,\end{aligned} \tag{2}$$

We identify $T_{\rm K}$ as the highest temperature for which Eqs. (2) have a non-trivial solution. Depending on the value of μ we can have two distinct situations. For $\mu = 0$, i.e. when the chemical potential of the 3D SM lies exactly at the DP, the first of Eq. (2) in the limit $\mu_f, |s|^2 \to 0$ gives

$$T_{\rm K} = D\frac{\sqrt{3}}{\pi}\sqrt{1-\frac{2}{\mathcal{N}(D)J}}\;, \qquad \mu = 0\;. \tag{3}$$

Eq. (3) is valid only for J larger than the critical value $J_{\rm cr} = 2/\mathcal{N}(D)$, and $T_{\rm K}$ vanishes when this condition is not met. This threshold-like behavior is well-known to occur [32, 33, 36, 37, 43, 49] when the DOS vanishes for $\varepsilon \to 0$. A similar situation is realized in 2D for which one obtains $T_{\rm K} = D\left[1-1/(\mathcal{N}(D)J)\right]/\ln(4)$ [32, 33, 37, 43, 50].

When $\mu \neq 0$, in the limit $k_{\rm B}T_{\rm K} \ll \mu \ll D$ and $J \lesssim J_{\rm c}$ we obtain

$$T_{\rm K} = D\exp\left[\frac{1-2/(J\mathcal{N}(D))}{2\mu^2/D^2}\right]\;, \qquad \mu \neq 0\;. \tag{4}$$

For $J \gtrsim J_{\rm c}$ is not possible to get a compact analytic expression for $T_{\rm K}$. In 2D [39] and for $J \lesssim J_{\rm cr}$ we have instead $T_{\rm K} = \kappa(\mu)e^{[1-1/(\mathcal{N}(D)J)]/|\mu/D|}$, where $\kappa(\mu) = \mu^2/D$ [$\kappa(\mu) = D$] for $\mu > 0$ [$\mu < 0$]. Fig. 1 shows the Kondo temperature $T_{\rm K}$ of 3D and 2D SMs as a function of J (both smaller and larger that $J_{\rm c}$) and for different values of $\mu > 0$, as derived from the solution of the self-consistent Eqs. (2).

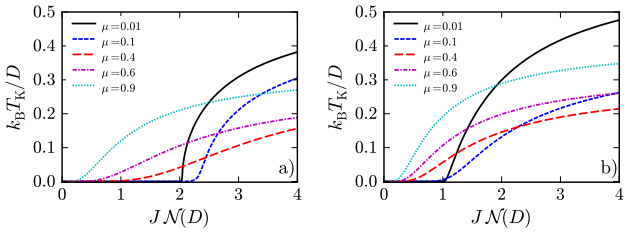

FIG. 1. (Color online) Panel a) the Kondo temperature of a 3D Weyl/Dirac material in units of half the bandwidth, plot as a function of the DOS at the bottom of the band $\mathcal{N}(D)$ and for several value of the chemical potential μ. Panel b) same as in panel a) but for a 2D system.

The coupling term H_J induces a renormalization of the Green's function of the SM electrons. Let

$$G^{(f)}_{\sigma\sigma'}(i\omega_m) = \delta_{\sigma\sigma'}\left[i\omega_m - \mu_f - |s|^2\sum_{\boldsymbol{k},j} G^{(0,j)}_{\sigma\sigma}(\boldsymbol{k},i\omega_m)\right]^{-1} \tag{5}$$

be the Green's function of the f-electrons, with $G^{(0,j)}_{\sigma\sigma'}(\boldsymbol{k},i\omega_m)$ the Green's function of electrons of the SM in the clean limit, and $\omega_n = \pi T(2n+1)$ the fermionic Matsubara frequencies. In Eq. (5) the sum is extended to all the wavevectors $\boldsymbol{k}$ with $|\boldsymbol{k}| < D/(\hbar v_{\rm F})$, and to all the DPs $j = 1,\ldots,N_{\rm w}$ in the Brillouin zone. Note that only the diagonal part of $G^{(0,j)}_{\sigma\sigma'}(\boldsymbol{k},i\omega_m)$ survives the summation, and that μ_f plays the role of the chemical potential

of the f-electrons. The renormalized Green's function of the itinerant electrons is given by

$$G^{(j)}_{\sigma\sigma'}(\boldsymbol{k}, i\omega_m) = G^{(0,j)}_{\sigma\sigma'}(\boldsymbol{k}, i\omega_m) + |s|^2 \sum_{\sigma''} G^{(0,j)}_{\sigma\sigma''}(\boldsymbol{k}, i\omega_m) G^{(f)}_{\sigma''\sigma''}(i\omega_m) G^{(0,j)}_{\sigma''\sigma'}(\boldsymbol{k}, i\omega_m) \ . \tag{6}$$

From this expression it is immediate to find the relaxation time $\tau(\varepsilon)$ of the electrons in the SM. We recall that $1/\tau(\varepsilon)$ is proportional to the imaginary part of the T-matrix which, as shown in Eq. (6), is proportional to $\mathrm{Im}[G^{(f)}_{\sigma\sigma}(\varepsilon+i\eta)]$. We get $1/\tau(\varepsilon) = -2n_{\rm imp}|s|^2\mathrm{Im}[G^{(f)}_{\sigma\sigma}(\varepsilon+i\eta)] = 4n_{\rm imp}/\left[\pi\mathcal{N}(\varepsilon+\mu)\right]$. It is interesting to point out that $\tau(\varepsilon)$ does not depend on the hybridization $|s|^2$. Indeed the factor $|s|^2$, due to the interaction vertices between electrons and impurity states, is canceled by an opposite factor $\sim 1/|s|^2$ due to the peak of $\mathrm{Im}[G^{(f)}(\omega)]$ at the Fermi energy. Essentially, even though the electron-impurity coupling becomes stronger by increasing s, its effect is compensated by the reduced spectral weight of impurity states at the Fermi energy. Using Boltzmann-transport theory and the expression of $\tau(\varepsilon)$, we can estimate the zero-temperature Kondo resistivity (restoring $\hbar$)

$$\rho_{\rm K}(T=0) = \frac{h}{e^2}\left(\frac{32 g_{\rm s}}{3\pi^2 N_{\rm w}^2}\right)^{1/3} \frac{n_{\rm imp}}{n^{4/3}} \ . \tag{7}$$

It is interesting to compare the scalings of $\rho_{\rm K}$ and of the electrical resistivity ρ (due to non-magnetic disorder) with the density. We recall that in the case of scalar short-range impurities [3] ρ is independent of n. The scaling of $\rho_{\rm K}$ given in Eq. (7) coincides with that of ρ when the latter is caused by charged impurities [3]. This is not a coincidence, since $\tau(0)$ scales with the DOS, the charge transport time is proportional to $\left[\mathcal{N}(\mu)u^2(0)\right]^{-1}$, and the screened Coulomb potential $u(q)$ of impurities scales with the inverse of the DOS. The same happens also in the 2D case, for which the zero-temperature Kondo resistivity is [38] $\rho_K = (h/e^2)[4n_{\rm imp}/(\pi N_{\rm w})]n^{-1}$.

So far we have considered the effect of magnetic impurities in a clean SM. However, some amount of charged impurities is always present in any experimental sample. In Dirac SMs, differently from "standard" metals, charged impurities induce [51–53] long-range carrier density inhomogeneities [54, 55]. Such inhomogeneities have been observed in direct imaging experiments in graphene [56–58] and TIs [24, 59, 60]. Since the DOS of 3D Dirac SMs scales with the density as $\sim n^{2/3}$, the long-range fluctuations of the carrier density reflect on the DOS and, as it is shown by Eq. (4), on the Kondo temperature $T_{\rm K}$. The Kondo effect is not characterized anymore by a single value of $T_{\rm K}$, but by a distribution of Kondo temperatures $P(T_{\rm K})$ [44]. A similar situation was predicted to occur in metals close to a metal-insulator transition (MIT) [28].

To study the interplay of Kondo screening and long-range disorder we consider a Gaussian density distribution $P_n(n)$ centered around the average doping $\bar{n}$, with standard deviation σ_n (proportional to the number of dopants), *i.e.* $P_n(n) = \exp\left[-(n-\bar{n})^2/(2\sigma_n^2)\right]/(\sqrt{2\pi}\sigma_n)$. This assumption for $P_n(n)$ has been shown to be well justified for the case of 2D graphene [61–63] and we expect it to be a reasonable model also for 3D SMs. Using this expression for $P_n(n)$ and the fact that $\mu \sim n^{1/3}$, from Eq. (4) we obtain

$$P^{(\rm 3D)}(T_{\rm K}) = \frac{3D^3}{8\sqrt{\pi}\sigma_\mu^3 T_{\rm K}}\left[\frac{\left(1-J_{\rm c}/J\right)^3}{\ln^5(k_{\rm B}T_{\rm K}/D)}\right]^{1/2} \times \left[e^{-\frac{(\mu^3-\bar{\mu}^3)^2}{2\sigma_\mu^6}} + e^{-\frac{(\mu^3+\bar{\mu}^3)^2}{2\sigma_\mu^6}}\right] , \tag{8}$$

where $\bar{\mu} = v_{\rm F}(6\pi^2\bar{n}/N_{\rm w})^{1/3}$, $\sigma_\mu = v_{\rm F}(6\pi^2\sigma_n/N_{\rm w})^{1/3}$, and $\mu \equiv \mu(T)$ is obtained by inverting Eq. (4).

We recall that in 2D $|\mu| \sim n^{1/2}$. The major complication in this case is due to the asymmetric prefactor $\kappa(\mu)$ of the exponential in the expression of $T_{\rm K}$ [see discussion after Eq. (4)], which we approximate with the half-bandwidth D. In this way we obtain a lower bound for the distribution of Kondo temperature $P^{(\rm 2D)}(T_{\rm K})$. Inverting the expression for the Kondo temperature we get $\mu_{\rm 2D}(T) \sim D(1-J_{\rm c}/J)/\ln(k_{\rm B}T/D)$ and, in the limit of $T_{\rm K} \to 0$,

$$P^{(\rm 2D)}(T_{\rm K}) = \frac{\sqrt{2}D^2}{\sqrt{\pi}\sigma_\mu^2 T_{\rm K}}\frac{(1-J_{\rm c}/J)^2}{|\ln^3(k_{\rm B}T_{\rm K}/D)|} \times \left[e^{-\frac{(\mu^2-\bar{\mu}^2)^2}{2\sigma_\mu^4}} + e^{-\frac{(\mu^2+\bar{\mu}^2)^2}{2\sigma_\mu^4}}\right] , \tag{9}$$

where in this expression $\mu \equiv \mu_{\rm 2D}(T)$.

Eqs. (8) and (9) show explicitly that the distribution of Kondo temperatures behaves, in the limit $T_{\rm K} \to 0$, as

$$P^{(\rm 3D)}(T_{\rm K}) \propto T_{\rm K}^{-1}|\ln(T_{\rm K})|^{-5/2} e^{-\bar{\mu}^6/(2\sigma_\mu^6)} \ , \\ P^{(\rm 2D)}(T_{\rm K}) \propto T_{\rm K}^{-1}|\ln(T_{\rm K})|^{-3} e^{-\bar{\mu}^4/(2\sigma_\mu^4)} \ . \tag{10}$$

Thus, in the presence of long-range disorder there is always a large fraction of the sample whose Kondo temperature is extremely small. As a consequence at any finite temperature a significant fraction of carriers is not "bound" to the magnetic impurities. From Eqs. (9) and (8) we determine the number of free spins as $n_{\rm fr}(T) = \int_0^T dT_{\rm K} P(T_{\rm K})$ and in the limit of $T \to 0$ we find

$$n_{\rm fr}(T) \propto |\ln(T)|^{-3/2} e^{-\bar{n}^2/(2\sigma_n^2)} \quad \text{in 3D} \ , \\ n_{\rm fr}(T) \propto |\ln(T)|^{-2} e^{-\bar{n}^2/(2\sigma_n^2)} \quad \text{in 2D} \ . \tag{11}$$

Eqs. (8)-(11) are the central results of this work. Plots of the distribution of Kondo temperatures in 3D and 2D SMs are shown in Fig. 2.

Note that the number of free spins diverges logarithmically as $T \to 0$, and so does the magnetic susceptibility $\chi_{\rm m}(T) \propto n_{\rm fr}(T)/T$. At odds with the magnetic

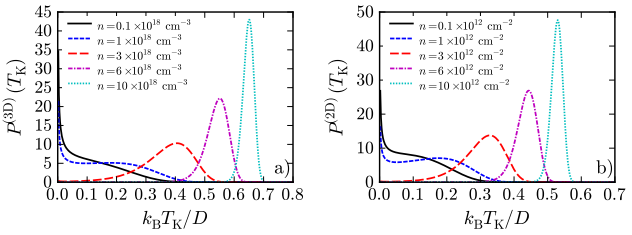

FIG. 2. (Color online) Panel a) the distribution of Kondo temperatures of a 3D Weyl/Dirac material plot as a function of the temperature in units of half the bandwidth and for several value of the excess carrier density. In this plot we set $\sigma_n = 10^{18}$ cm^{-3}, $v_F = 10^8$ cm/s, $J = 0.6\ J_c$, $N_w = 2$ and a half-bandwidth $D = 0.5$ eV. Panel b) same as in panel a) but for a 2D system. Here $\sigma_n = 10^{12}$ cm^{-2}, while all the other parameters coincide with those of panel a).

susceptibility of a normal Fermi liquid, $\chi_m(T)$ diverges slower than $\sim 1/T$ (Curie-Weiss law) and does not converge to any finite value at zero temperature [64]. This is a clear signature of the development of a NFL behavior. We observe that in Dirac SMs the divergence of $\chi_m(T)$ is stronger than what was found for metals close to a MIT [28]. Note also that both the distribution $P(T_K)$ and the number of free spins contain the factor $\exp\left[-\bar{n}^2/(2\sigma_n^2)\right]$, which encodes the effects of both doping and disorder. If the system is strongly doped (*i.e.* if $\bar{n} \gg \sigma_n$), the exponential factor strongly suppresses the NFL behavior. The density fluctuations are indeed too small and the Kondo effect is completely controlled by the average Kondo temperature $\langle T_K \rangle$. In this situation, the spin susceptibility diverges only at extremely small temperatures. On the contrary, when the density fluctuations are strong, *i.e.* when $\bar{n} \ll \sigma_n$, the exponential factor is of order of the unity, and the number of free spins can be quite large. We find that the number of free spins is $n_{fr} \sim 22$ % at $T = 30$ K in a 3D doped Dirac SM ($\bar{n} = 10^{16}$ cm^{-3}) with density fluctuations $\sigma_n = 10^{18}$ cm^{-3}. To obtain this estimate we used a Fermi velocity $v_F = 10^8$ cm/s, $J = 0.6\ J_c$, $N_w = 2$ and a bandwidth $D = 0.5$ eV. The parameters have been chosen to have $k_BT \ll \mu \ll D$.

In conclusion, we have studied the Kondo effect in 3D Dirac and Weyl semimetals. In the absence of long-range, disorder-induced, carrier density inhomogeneities the Kondo effect is characterized by the Kondo temperature T_K, the crossover temperature below which Kondo screening takes effect. When the chemical potential μ is pinned at the Dirac point we find that no Kondo effect can take place unless the coupling J between magnetic impurities and conduction electrons is larger than a critical value $J_c = 2/\mathcal{N}(D)$, in this case $T_K \propto \sqrt{1 - J_c/J}$. The existence of a critical coupling is analogous to the case of graphene and, in general, to the pseudogap Kondo problem. For $\mu > 0$, T_K is different from zero for any value of J and depends exponentially on μ and J. We also find that in the low-temperature regime ($T \to 0$) the Kondo resistivity due to the presence of magnetic impurities scales as $\rho_K \propto n_{imp}/n^{4/3}$.

In the presence of long-range disorder we find that the Kondo effect is not characterized by a single crossover temperature T_K, but by a distribution of Kondo temperatures $P(T_K)$, as a result of the disorder-induced carrier density inhomogeneities and the fact that, due to the linear dispersion, the density-of-states depends on the local value of the carrier density. We find that, in the limit of $T_K \to 0$, $P(T_K) \propto T_K^{-1} |\ln(T_K)|^{-5/2}$, and that the number of screened magnetic impurities goes to zero in the same limit. This in turn implies that the magnetic susceptibility diverges slower than $\sim 1/T$ for $T \to 0$, and that it does not converge to any finite value at zero temperature. As was pointed out in Ref. [28], this is the signature of a strong NFL behavior. We find a qualitatively similar behavior also in 2D. In this case $P(T_K) \propto T_K^{-1} |\ln(T_K)|^{-3}$. Transport measurements in graphene in the presence of vacancies have already observed the signature of Kondo effect [65]. It would be interesting to study in these samples the low-temperature behavior of quantities such as the magnetic susceptibility to verify the existence of the NFL behavior.

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