INTRODUCTION TO THE SPECIAL FUNCTIONS OF MATHEMATICAL PHYSICS

with applications to the physical and applied sciences

John Michael Finn

April 13, 2005

CONTENTS

PREFACE

This text is based on a one semester advanced undergraduate course that I have taught at the College of William and Mary. In the spring semester of 2005, I decided to collect my notes and to present them in a more formal manner. The course covers selected topics on mathematical methods in the physical sciences and is cross listed at the senior level in the physics and applied sciences departments. The intended audience is junior and senior science majors intending to continue their studies in the pure and applied sciences at the graduate level. The course, as taught at the College, is hugely successful. The most frequent comment has been that students wished they had been introduced to this material earlier in their studies.

Any course on mathematical methods necessarily involves a choice from a venue of topics that could be covered. The emphasis on this course is to introduce students the special functions of mathematical physics with emphasis on those techniques that would be most useful in preparing a student to enter a program of graduate studies in the sciences or the engineering disciplines. The students that I have taught at the College are the generally the best in their respective programs and have a solid foundation in basic methods. Their mathematical preparation includes, at a minimum, courses in ordinary differential equations, linear algebra, and multivariable calculus. The least experienced junior level students have taken at least two semesters of Lagrangian mechanics, a semester of quantum mechanics, and are enrolled in a course in electrodynamics, concurrently. The senior level students have completed most of their required course work and are well into their senior research projects. This allows me to exclude a number of preliminary subjects, and to concentrate on those topics that I think would be most helpful. My classroom approach is highly interactive, with students presenting several in-class presentations over the course of the semester. In-class discussion is often lively and prolonged. It is a pleasure to be teaching students that are genuinely interested and engaged. I spend significant time in discussing the limitation as well as the applicability of mathematical methods, drawing from my own experience as a research scientist in particle and nuclear physics. When I discuss computational algorithms, I try to do so .from a programming language-neutral point of view.

The course begins with review of infinite series and complex analysis, then covers Gamma and Elliptic functions in some detail, before turning to the main theme of the course: the unified study of the most ubiquitous scalar partial differential equations of physics, namely the wave, diffusion, Laplace, Poisson, and Schrödinger equations. I show how the same mathematical methods apply to a variety of physical phenomena, giving the students a global overview of the commonality of language and techniques used in various subfields of study. As an intermediate step, Strum-Liouville theory is used to study the most common orthogonal functions needed to separate variables in Cartesian, cylindrical and spherical coordinate systems. Boundary valued problems are then studied in detail, and integral transforms are discussed, including the study of Green functions and propagators.

The level of the presentation is a step below that of *Mathematical Methods for Physicists* by George B. Arfken and Hans J. Weber, which is a great book at the graduate level, or as a desktop reference; and a step above that of *Mathematical Methods in the Physical Sciences*, by Mary L. Boas, whose clear and simple presentation of basic concepts is more accessible to an undergraduate audience. I have tried to improve on the rigor of her presentation, drawing on material from Arfken, without overwhelming the students, who are getting their first exposure to much of this material.

Serious students of mathematical physics will find it useful to invest in a good handbook of integrals and tables. My favorite is the classic *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables* (AMS55), edited by Milton Abramowitz and Irene A. Stegun. This book is in the public domain, and electronic versions are available for downloading on the worldwide web. NIST is in the process of updating this work and plans to make an online version accessible in the near future.

Such handbooks, although useful as references, are no longer the primary means of accessing the special functions of mathematical physics. A number of high level programs exist that are better suited for this purpose, including Mathematica, Maple, MATHLAB, and Mathcad. The College has site licenses for several of these programs, and I let students use their program of choice. These packages each have their strengths and weaknesses, and I have tried to avoid the temptation of relying too heavily on proprietary technology that might be quickly outdated. My own pedagogical inclination is to have students work out problems from first principles and to only use these programs to confirm their results and/or to assist in the presentation and visualization of data. I want to know what my students know, not what some computer algorithm spits out for them.

The more computer savvy students might want to consider using a high-level programming language, coupled with good numeric and plotting libraries, to achieve the same results. For example, the Ch scripting interpreter, from SoftIntegration, Inc, is available for most computing platforms including Windows, Linux, Mac OSX, Solaris, and HP-UX. It includes high level C99 scientific math libraries and a decent plot package. It is a free download for academic purposes. In my own work, I find the C# programming language, within the Microsoft Visual Studio programming environment, to be suitable for larger web-oriented projects. The C# language is an international EMCA supported specification. The .Net framework has been ported to other platforms and is available under an open source license from the MONO project.

These notes are intended to be used in a classroom, or other academic settings, either as a standalone text or as supplementary material. I would appreciate feedback on ways this text can be improved. I am deeply appreciative of the students who assisted in this effort and to whom this text is dedicated.

Williamsburg, Virginia — John Michael Finn
April, 2005 — Professor of Physics
finn@physics.wm.edu

DEDICATION

For my students
at The College of
William and Mary
in Virginia

1. Infinite Series

The universe simply is.
Existence is not required to explain itself.
This is a task that mankind has chosen for himself,
and the reason that he invented mathematics.

1.1 Convergence

The ancient Greeks were fascinated by the concept of infinity. They were aware that there was something transcendental beyond the realm of rational numbers and the limits of finite algebraic calculation, even if they did not fully comprehend how to deal with it. Some of the most famous paradoxes of antiquity, attributed to Zeno, wrestle with the question of convergence. If a process takes an infinite number of steps to calculate, does that necessarily imply that it takes an infinite amount of time? One such paradox purportedly demonstrated that motion was impossible, a clear absurdity. Convergence was a concept that mankind had to master before he was ready for Newton and his calculus.

Physicists tend to take a cavalier attitude to convergence and limits in general. To some extent, they can afford to. Physical particles are different than mathematical points. They have a property, called inertia, which limits their response to external force. In the context of special relativity, even their velocity remains finite. Therefore, physical trajectories are necessarily well-behaved, single-valued, continuous and differentiable functions of time from the moment of their creation to the moment of their annihilation. Mathematicians should be so fortunate. Nevertheless, physicists, applied scientists, and engineering professionals cannot afford to be too cavalier in their attitude. Unlike the young, the innocent, and the unlucky, they need to be aware of the pitfalls that can befall them. Mathematics is not reality, but only a tool that we use to image reality. One needs to be aware of its limitations and unspoken assumptions.

Infinite series and the theory of convergence are fundamental to the calculus. They are taught as an introduction to most introductory analysis courses. Those who stayed awake in lecture may even remember the proofs—Therefore, this chapter is intended as a review of things previously learnt, but perhaps forgotten, or somehow neglected. We begin with a story.

1.2 A cautionary tale

The king of Persia had an astronomer that he wished to honor, for just cause. Calling him into his presence, the king said that he could ask whatever he willed, and, if it were within his power to grant it, even if it were half his kingdom, he would.

To this, the astronomer responded: “O King, I am a humble man, with few needs. See this chessboard before us that we have played on many times, grant me only one gain of gold for the first square, and if it please you, twice that number for the second square, and twice that again for the third square, and so forth, continuing this pattern, until the board is complete. That would be reward enough for me.” The king was pleased at such a modest request, and commanded his money changer to fulfill the astronomer’s wish. Figure 1-1 shows the layout of the chessboard, and gives some inkling of where the calculation may lead.

					…	2^{62}	2^{63}
1	2	2^2	2^3	2^4	2^5	…	

Figure 1-1 Layout of the King’s chessboard.

The total number grains of gold is a sequence whose sum is given by S=1+2+2^2+2^3+2^4+..., or more generally

$$S = \sum_{n=0}^{63} 2^n \; . \tag{1.1}$$

Note that mathematicians like to start counting at zero since $x^0 = 1$ is a good way to include a leading constant term in a power series. Many present day computer programs number their arrays starting at zero for the first element as well.

The above is an example of a finite sequence of numbers to be summed, a series of N terms, defined by an , which can be written as

$$S_N = \sum_{n=0}^{N-1} a_n \; , \tag{1.2}$$

where a_n denotes the n^{th} element in the sum of a series of N terms, expressed as S_N. The algorithm or rule for defining the constants in our chess problem is given by the prescription

$$a_0 = 1, \;\; \text{and} \;\; a_{n+1} = 2a_n \; . \tag{1.3}$$

Note that a is used to compactly describe the progression. For infinite series, where it is physically impossible to write down every single term, the series must be defined by such a rule, often recursively derived, for constructing the n^{th} term in the series.

Most of us are familiar with computers and know that they store data in binary format (see Figure 1-2). A bit set in the n^{th} place represents the number 2^n. Our chess problem corresponds to a binary number with the bit pattern of a 64 bit integer have all its bits set, the largest unsigned number that can be stored in 64 bits. Adding one to this number results in all zeroes plus the setting of an overflow bit representing the number 2^{64}. Therefore, the answer to our chess problem would require ($2^{64}-1$) grains of gold. This is a huge number, considering that there are there are only $6.02 \cdot 10^{23}$ atoms per gram-mole of gold.

11111111 11111111 11111111 11111111 11111111 11111111 11111111 11111111

Figure 1-2 A 64-bit unsigned-integer bit-pattern with all its bits set

I could continue the story to its conclusion, but it is more interesting to leave you to speculate as to possible outcomes. Here are some questions to ponder:

- What do you suppose the king did to the astronomer? Was this something to lose one's head over?
- Most good stories have a point, a moral, or a lesson to be learnt. What can one learn from this story?
- If N goes to infinity does the series converge? If not, why not?

(Hint: Preliminary test: if the terms in the series an do not tend to zero as $n \to \infty$, the series diverges)

1.3 Geometric series

The chess board series is an example of a , one where successive terms are multiplied by a constant ratio *r*. It represents one of the oldest and best known of series. A geometric series can be written in the general form as

$$G_N\left(a_0,r\right)=a_0(1+r+r^{,2}+\ldots)=a_0\sum_0^{N-1}r^n\ . \tag{1.4}$$

The initial term is $a_0=1$ and the ratio is $r=2$ for the chess board problem. The sum of a geometric series can be evaluated giving solution with a closed form:

$$G_N(a_0,r)=a_0\sum_0^{N-1}r^n=a_0\left(\frac{1-r^n}{1-r}\right). \tag{1.5}$$

❖ Proof by mathematical induction

There are a number of ways that the formula for the sum a geometric series (1.5) can be verified. Perhaps the most useful for future applications to other recursive problems is Induction involves carrying out the following three logical steps.

- Verify that the expression is true for the initial case. Letting $a_0=1$ for simplicity, one gets

$$G_1\left(r\right)=\left(\frac{1-r^1}{1-r}\right)=1. \tag{1.6}$$

- Assume that the expression is true for the N^{th} case, i.e., assume

$$G_N(r) = \left(\frac{r^N - 1}{r - 1} \right). \tag{1.7}$$

- Prove that it is true for the $N+1$ case:

$$\begin{aligned}
G_{N+1} &= G_N + r^N = \left(\frac{r^N - 1}{r - 1} \right) + r^N, \\
G_{N+1} &= \left(\frac{r^N - 1}{r - 1} \right) + r^N \left(\frac{r - 1}{r - 1} \right) = \frac{r^N - 1 + r^N (r - 1)}{r - 1}, \\
G_{N+1} &= \frac{r^N - 1 + r^{N+1} - r^N)}{r - 1} = \left(\frac{r^{N+1} - 1}{r - 1} \right) = \left(\frac{1 - r^{N+1}}{1 - r} \right).
\end{aligned} \tag{1.8}$$

1.4 Definition of an infinite series

The sum of an infinite series $S(r)$ can be defined as the sum of a series of N terms $S_N(r)$ in the limit as the number of terms N goes to infinity. For the geometric series this becomes

$$G(r) = \lim_{N \to \infty} G_N(r) \tag{1.9}$$

or

$$G(r) = G_\infty(r) = \left(\frac{1 - r^\infty}{1 - r} \right), \tag{1.10}$$

where

$$r^{\infty} \to \begin{cases} 0 & if \;\; r<1, \\ \infty & f \;\; r>1, \\ 1+1+\ldots & diverges. \end{cases} \tag{1.11}$$

Therefore, for a general infinite geometric series,

$$G(a_0, r) = \begin{cases} \sum_{n=0}^{\infty} a_0 r^n = \dfrac{a_0}{1-r} & for \; r<1, \\ \text{undefined for } r \geq 1. \end{cases} \tag{1.12}$$

❖ Convergence of the chessboard problem

Let's calculate how much gold we could obtain if we had a chessboard of infinite size. First, let's try plugging into the series solution:

$$\begin{aligned} S_N &= \sum_{n=0}^{N-1} 2^n = G_N(1,2) = \frac{2^N - 1}{2-1}, \\ S_N &\underset{N\to\infty}{\to} \frac{1}{1-2} = -1. \end{aligned} \tag{1.13}$$

This is clearly nonsense. One can not get a negative result by adding a sequence that contains only positive terms. Note, however, that the series converges only if $r<1$. This leads to our first two major conclusions:

- The sum of a series is only meaningful if it converges.
- A function and the sum of the power series that it represents are mathematically equivalent within, and only within, the radius of convergence of the power series.

❖ Distance traveled by A bouncing ball

Here is an interesting variation on one of Zeno's Paradoxes: If a bouncing ball bounces an infinite number of times before coming to rest, does this necessarily imply that it will bounce forever? Answers to questions like this led to the development of the formal theory of convergence. The detailed definition of the problem to be solved is presented below.

Discussion Problem: A physics teacher drops a ball from rest at a height h_0 above a level floor. See Figure 1-3. The acceleration of gravity g is constant. He neglects air resistance and assumes that the collision, which is inelastic, takes negligible time (using the impulse approximation). He finds that the height of each succeeding bounce is reduced by a constant ratio r, so $h_n = rh_{n-1}$

- Calculate the total distance traveled, as a Geometric series.
- Using Newton's Laws of Motion, calculate the time t_n needed to drop from a height h_n.
- Write down a series for the total time for N bounces. Does this series converge? Why or why not?

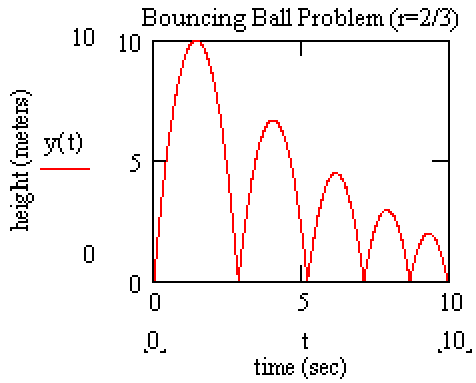


Figure 1-3 Height (m) vs. time (s) for a bouncing ball

Figure 1-3 shows a plot of the motion of a bouncing ball (h_0=10 m, r=2/3, g=9.8 m/s^2). The height of each bounce is reduced by a constant ratio. A complication is that the total distance traveled is to be calculated from the maximum of the first cycle, requiring a correction to the first term in the series.

The series to be evaluated turn out to be geometric series. The motion of the particle for the first cycle is given by

$$\begin{aligned} y(t) &= v_0 t - \frac{gt^2}{2}, \\ v_0 &= \sqrt{2gh_0}, \\ \Delta t_0 &= 2v_0 / g = \sqrt{8h_0 / g}, \end{aligned} \tag{1.14}$$

where Δt_0 is the time for the first cycle, and

$$D = \sum 2h_n - h_0 = 2h_0 \sum r^n - h_0 = \frac{2h_0}{1-r} - h_0,$$
$$\Delta t_n = \sqrt{8h_0 r / g} = \Delta t_0 \sqrt{r}, \tag{1.15}$$
$$T = \frac{\Delta t_0}{1-\sqrt{r}} - \frac{\Delta t_0}{2}.$$

Both series converge, but the time series converges slower than the distance series as illustrated in Figure 1-4 below, since $\sqrt{r} > r$ for $r < 1$.

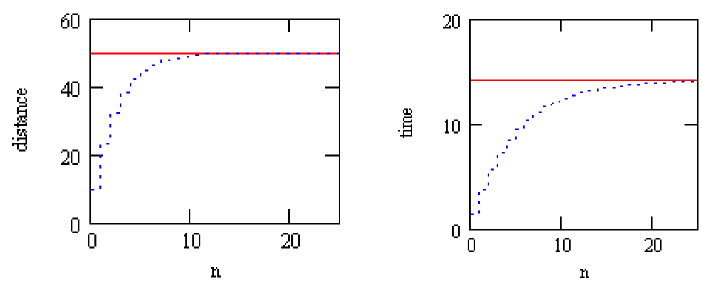


Figure 1-4 The distance and time traveled by a bouncing ball (h_0=10 m, g=9.8 m/s^2 and r=2/3)

The bouncing ball undergoes an infinite number of bounces in a finite time. For a contrary example, an under-damped oscillator undergoes an infinite number of oscillations and requires an infinite amount of time to come to rest.

1.5 The remainder of a series

An infinite series can be factored into two terms

$$S = S_N + R_N, \tag{1.16}$$

where

- $S_N = \sum_{n=0}^{N-1} a_n$ is the , which has a finite sum of terms, and
- R_N is the of the series, which has an infinite number of terms

$$R_N = \sum_{n=N}^{\infty} a_n \tag{1.17}$$

1.6 Comments about series

- S_N denotes the partial sum of an infinite series S, the part that is actually calculated. Since its computation involves a finite number of algebraic operations (i.e., it is a finite algorithm) computing it poses no conceptual challenge. In other words, the rules of algebra apply, and a program can happily be written to return the result.
- The remainder of a series S, denoted as R_N, is an infinite series. This series may, or may not, converge.
- The convergence of R_N is the same as the convergence of the series S. The convergence of an infinite series is not affected by the addition or subtraction of a finite number of leading terms.

- Computers (and humans too) can only calculate a finite number of terms, therefore an estimate of R_N is needed as a measure of the error in the calculation.
- The most essential component of an infinite series is its remainder—the part you don't calculate. If one can't estimate or bound the error, the numerical value of the resulting expression is worthless.

Before using a series, one needs to know

- whether the series converges,
- how fast it converges, and
- what reasonable error bound one can place on the remainder R_N.

1.7 The Formal definition of convergence

$$\begin{gathered}\text{A series } S = S_N + R_N \text{ converges if} \\ |R_N| = |S - S_N| < \varepsilon \quad \text{for all } N > N(\varepsilon).\end{gathered} \tag{1.18}$$

1.8 Alternating series

Definition: *A series of terms with alternating terms is an alternating series*

Consider the alternating series A, given by

$$A = \sum_n (-1)^n |a_n| \tag{1.19}$$

An alternating series converges if

$$\begin{aligned} &a_n \to 0, \quad \text{as } n \to \infty, \text{ and} \\ &|a_{n+1}| < |a_n|, \quad \text{for all } \; n > \mathrm{N}_0 \end{aligned} \tag{1.20}$$

An oscillation of sign in a series can greatly improve its rate of convergence. For a series of reducing terms, it is easy to define a maximum error in a given approximation. The error in S_N is smaller that the first neglected term

$$|R_N| < a_N . \tag{1.21}$$

❖ Alternating Harmonic Series

An alternating harmonic series is defined as the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots . \tag{1.22}$$

This is a decreasing alternating series which tends to zero and so meets the preliminary test.

Example: Series expansion for the natural logarithm

In a book of math tables one can lookup the series expansion of the natural logarithm, which is

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{-(-x)^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x}{3} \cdots \tag{1.23}$$

Setting $x = 1$, allows one to calculate the sum of an alternating harmonic series in closed form

$$\ln(2) = \ln(1+1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 0.6931471806 \,. \tag{1.24}$$

Finding a functional representation of a series is a useful way of expressing its sum in closed form.

What about $\ln(0)$?

$$\begin{aligned} &\ln(1-1) = \sum_{n=0}^{\infty} \frac{1}{n+1} == 1 + \frac{1}{2} + \frac{1}{3} \cdots \\ &\ln(0) = -\infty \quad \textit{diverges} \end{aligned} \tag{1.25}$$

So, to summarize:

$$\begin{aligned} &S = \sum \left(\frac{(-1)^n}{n+1} \right) \quad \textit{converges} \\ &\textit{but} \\ &S = \sum \left| \frac{(-1)^n}{n+1} \right| \quad \textit{diverges} \end{aligned} \tag{1.26}$$

(1.27)

The series expansion for $\ln(1+x)$ can be summarized as

$$\boxed{\begin{aligned} &\ln(1+x) = \sum_{n=0}^{\infty} \frac{-(-x)^{n+1}}{n+1}, \\ &\textit{for} \quad 0 < x \le 2. \end{aligned}} \tag{1.28}$$

This series is only conditionally convergent at its end points, depending on the signs of the terms. What is going on here?

Let's rearrange the terms of the series so all the positive terms come first (real numbers are commutative aren't they?)

$$
\begin{aligned}
&S \to S^{+} + S^{-}, \\
&S^{+} = \sum_{n=0}^{\infty} \frac{1}{2n} \to \infty, \\
&S^{-} = -\sum_{n=0}^{\infty} \frac{1}{2n+1} \to -\infty, \\
&S = \infty - \infty \quad \text{is undefined.}
\end{aligned}
\tag{1.29}
$$

The problem is that a series is an algorithm, the first series diverges so one never gets around to calculating the terms of the second series (remember we are limited to a finite number of calculations, assuming we have only finite computer power available to us)

- Note that infinity is not a real number!

1.9 Absolute Convergence

A series converges absolutely if the sum of the series generated by taking the absolute value of all its terms converges. Let $S = \sum a_n$, then if the corresponding series of positive terms

$$S' = \sum |a_n| \tag{1.30}$$

converges, the initial series S is said to be Otherwise if S is convergent, but not absolutely convergent, it is said to be

Discussion Problem: Show that a conditional convergent series S can be made to converge to any desired value.

Here is an outline of a possible proof: Separate S into two series, one of which contains only positive terms and the second only negative terms. Since S is convergent, but not absolutely convergent, each of these series is separately divergent. Now borrow from the positive series until the sum is just greater than the desired value (assuming it is positive). Next subtract from the series just enough terms to bring it to just below the desired value. Repeat the process. If one has a series of decreasing terms, tending to zero, the results will oscillate about and eventually settle down to the desired value.

What is happening here is easy enough to understand: One can always borrow from infinity and still have an infinite number in reserve to draw upon:

$$\infty \pm a_n = \infty \tag{1.31}$$

You can simply mortgage your future to get the desired result.

Some conclusions:

- Conditionally convergent series are dangerous, the commutative law of addition does not apply (infinity is not a number).
- Absolutely convergent series always give the same answer no matter how the terms are rearranged.

- Absolute convergence is your friend. Don't settle for anything less than this. Absolutely convergent series can be treated as if they represent real numbers in an algebraic expression (they do). They can be added, subtracted, multiplied and divided with impunity.

❖ Distributive Law for scalar multiplication

A series can be multiplied term by term by a real number r without affecting its convergence

❖ Scalar multiplication

Scalar multiplication of a series by a real number r is given by

$$r\sum a_n = \sum r a_n. \tag{1.32}$$

❖ Addition of series

Two absolutely convergent series can be added to each other term by term; the resulting series converges within the common interval of convergence of the original series.

$$\begin{aligned} &\sum a_n \pm \sum b_n = \sum (a_n \pm b_n), \\ &c_1 S_a \pm c_2 S_b = \sum c_1 a_n \pm c_2 b_n. \end{aligned} \tag{1.33}$$

1.10 Tests for convergence

Here is a summary of a few of the most useful tests for convergence. Advanced references will list many more tests.

❖ Preliminary test

A series S_N diverges if its terms do not tend to zero as N goes to infinity.

❖ Comparison tests

Comparison tests involve comparing a series to a known series. Comparison can be used to test for convergence or divergence of a series:

- Given an absolutely convergent series $S_a = \sum a_n$ the series $S_b = \sum b_n$ converges if

$$|b_n| < |a_n| \tag{1.34}$$

  for $n > N$.

- Given an absolutely divergent series $S_a = \sum a_n$, the series $S_b = \sum b_n$ diverges if $|b_n| > |a_n|$ for $n > N$.

- Given a absolutely convergent series $S_a = \sum a_n$, the series $S_b = \sum b_n$ converges if

$$\left|\frac{b_{n+1}}{b_n}\right| < \left|\frac{a_{n+1}}{a_n}\right| \tag{1.35}$$

For $n > N$. (This test can also be used to test for divergence)

❖ The Ratio Test

This is a variant of the comparison test where the ratio of terms is compared to the geometric series: By comparison to a geometric series, the series $S_b = \sum b_n$ converges if the ratio of succeeding terms decreases as $n \to \infty$.

$$\begin{aligned} &\text{define } r = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| \\ &if \begin{cases} r < 1 & \text{the series converges,} \\ r > 1 & \text{the series diverges,} \\ r = 0 & \text{the test fails.} \end{cases} \end{aligned} \tag{1.36}$$

❖ The Integral Test

The series $S_b = \sum b_n$ converges if the upper limit of the integral obtained by replacing $\sum_{N_0}^{N} b_n \to \int_{N_0}^{N} b(n)dn$ converges as $N \to \infty$, and it diverges if the integral diverges. The proof is demonstrated graphically in Figure 1-5, which demonstrates that a sum of positive terms is bounded both above and below by its integral. The integral can be constructed to pass through all the steps in the partial sums at either the beginning or the end of an interval.

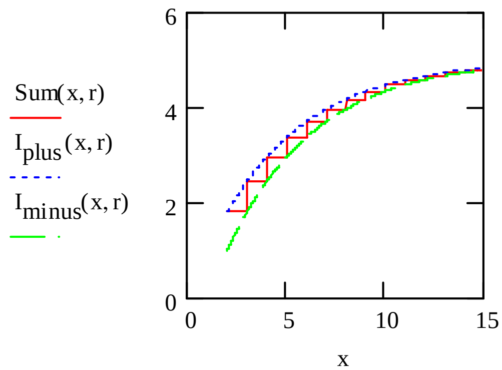


Figure 1-5 The Integral test

1.11 Radius of convergence

The series

$$S(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1.37}$$

defines a an absolutely convergent power series of x within its radius of convergence given by

$$\begin{aligned} r &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} x \right| < 1 \\ or \quad |x| &< \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|. \end{aligned} \tag{1.38}$$

Within its radius of convergence, the function and its power series are identical. The power series expansion of $S(x)$ is unique.

Example: Definition of the exponential function.

The exponential function is defined as that function which is its own derivative

$$\frac{de(x)}{dx} = e(x). \tag{1.39}$$

Let's show that the series expansion for the exponential function obeys this rule:

$$\begin{aligned}
&e^x = \sum_{n=0}^{\infty} a_n x^n \\
&\frac{de^x}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1} \\
&\textit{letting } n \to n' + 1 \text{ on the RHS} \\
&\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n'=0}^{\infty} (n'+1) a_{n'+1} x^{n'} = e^x \\
&\sum_{n'=0}^{\infty} (n'+1) a_{n'+1} x^{n'} = \sum_{n=0}^{\infty} a_n x^n.
\end{aligned} \tag{1.40}$$

Test for radius of convergence:

$$\begin{aligned}
&|x| < \lim \frac{(n+1)!}{n!} = (n=1) \\
&|x| < \infty.
\end{aligned} \tag{1.41}$$

(In practice, the useful range for computation is limited, depending on the format and storage allocation of a real variable in one's calculator.)

❖ Evaluation techniques

- if $|x| \leq 1$ this series is converges rapidly
- if $f\ x > 1$ use $e^{a+b} = e^a e^b$ to show

$$\begin{aligned} e^x &= \frac{1}{e^{-x}}, \\ e^{-x} &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}. \end{aligned} \tag{1.42}$$

Alternating series converge faster, it is easy to estimate the error, and if the algorithm is written properly one shouldn't get overflow errors. Professional grade mathematical libraries would use sophisticated algorithms to accurately evaluate a function over its entire useful domain.

1.12 Expansion of functions in power series

The power series of a function is unique within its radius of curvature. Since power series can be differentiated, we can use this property to extract the coefficients of a power series. Let us expand a function of the real variable x about the origin:

$$
\begin{aligned}
&f(x)=\sum_{n=0}^{\infty}a_n x^n; \\
&\textit{define} \\
&f^{(n)}(0)=\frac{d^n}{dx^n}f(x)\Big|_{x=0}; \\
&\textit{then} \\
&f(0)=a_0, \\
&f^{(1)}(0)=a_1, \\
&f^{(2)}(0)=2a_2, \\
&f^{(n)}(0)=n!a_n.
\end{aligned}
\tag{1.43}
$$

This results in the famous Taylor series expansion:

$$
f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n. \tag{1.44}
$$

Substituting $x \to x-a$ a, we get the generalization to a McLauren Series

$$
f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x-a)}{n!}(x-a)^n. \tag{1.45}
$$

Taylor's expansion can be used to generate many well known series, such as the exponential function and the binomial expansion.

❖ The binomial expansion

The binomial expansion is given by

$$\left(1+x\right)^p = \sum_{n=0}^{\infty} B(p,m)x^n \quad |x|<1, \tag{1.46}$$

where p is any real number. For integer p, the series is a polynomial with $p+1$ terms. The coefficients of this series are known as the binomial coefficients and written as

$$B(n,m) = \binom{p}{n} = \binom{p}{p-n} = \frac{p!}{(p-n)!n!}. \tag{1.47}$$

For non-integer p, but integer m, the coefficients can be expressed as the repeated product

$$B(n,m) = \frac{1}{n!}\prod_{m=0}^{n-1}(p-m). \tag{1.48}$$

❖ Repeated Products

occur often in solutions generated by iteration. A repeated product of terms r_m is denoted by the expression

$$\prod_{m=0}^{N-1} r_m = r_0 \cdot r_1 \cdot r_3 \cdots r_{N-1}. \tag{1.49}$$

The is an example of a repeated product

$$n! = \prod_{m=1}^{n} m. \tag{1.50}$$

Discussion Problem: Sine and cosine series

Euler's theorem, given by

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1.51}$$

is counted among the most elegant of mathematical equations.

Derive the series expansion of $\sin x$ and $\cos x$ using the power series expansion for e^x and substituting $x \to ix$, giving

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad |x| < \infty \tag{1.52}$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad |x| < \infty. \tag{1.53}$$

Then use the definition of the exponential, as the function which is its own derivative ($de^x / dx = e^x$) to prove

$$\begin{aligned} \frac{d \sin x}{dx} &= \cos x, \\ \frac{d \cos x}{dx} &= -\sin x. \end{aligned} \tag{1.54}$$

1.13 More properties of power series

- Power Series can be added, subtracted, and multiplied within their common radii of convergence. The result is another power series.
- Power Series can also be divided, but one needs to avoid division by zero. This may restrict the radius of convergence of the result.

- Power series can be substituted into each other to generate new power series. For example, one can substitute $x \to -x^2$ into the exponential function to get the power series of a Gaussian function:

$$e^{\left(-x^2\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}, \quad \left|x^2\right| < \infty. \tag{1.55}$$

1.14 Numerical techniques

Example: Calculating the series for $\ln(1+x)$ using long division

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}.$$

By long division:

$$\left(1+x\right)\overline{\big)1}^{\,1-x+x^2-} = \sum_{n=0}^{\infty}(-x)^n, \tag{1.56}$$

$$\therefore \ln(1+x) = \int_0^x \sum_{n=0}^{\infty}(-x)^n dx = \sum_{n=0}^{\infty} \frac{-\left(-x\right)^{n+1}}{n+1}.$$

Example: Evaluation of indeterminate forms by series expansion:

$$\lim_{x\to 0}\frac{1-e^{-x}}{x} = \frac{1}{x}\left(1-\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) = 1-\left(1-x+\frac{x^2}{2}+\ldots\right),$$

$$= \frac{1}{x}\sum_{n=0}^{\infty}\frac{x\left(-x\right)^n}{n+1!} = \sum_{n=0}^{\infty}\frac{\left(-x\right)^n}{n+1!}, \tag{1.57}$$

1.15 Series solutions of differential equations

The equation

$$\sum_{i=0}^{N} A_i(x)\left(\frac{d}{dx}\right)^i y(x) = S(x) \tag{1.58}$$

defines an N^{th} order linear differential equation for $y(x)$. If the source term $S(x)=0$, the equation is said to be homogeneous. Otherwise, the equation is said to be inhomogeneous. We will concern ourselves with solutions to homogeneous equations at first. A linear homogenous equation has the general form

$$\sum_{i=0}^{N} A_i(x)\left(\frac{d}{dx}\right)^i y(x) = 0 \tag{1.59}$$

A N^{th} order differential equation has N linearly-independent solutions $\{y_i(x)\}$, and by linearity, the general solution can be written as

$$y(x) = \sum_{i=0}^{N} c_i y_i(x) \tag{1.60}$$

If the coefficients $A_i(x)$ can be expanded in a power series about the point $x=0$, one can attempt to solve for $y_i(x)$ in terms of a power series expansion of the form

$$y_i(x) = \sum_{n=0}^{\infty} a_{im} x^n \tag{1.61}$$

Since the function is linear in y, the resulting series expansion will be linear in the coefficients a_{in}, and the self-consistent solution will involve recursion relations between the coefficients of various powers of m.

❖ A simple first order linear differential equation

Consider the first order differential equation

$$\frac{dY(x)}{dx} = -Y(x) \tag{1.62}$$

We already know the solution, it is given by

$$Y(x) = Y_0 e^{-x} \tag{1.63}$$

Since the equation is of first order, there is only one linearly independent solution so the above solution is complete. Let's try expanding this function in a power series

$$Y(x) = \sum_{n=0}^{\infty} a_n x^n; \quad Y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n'=0}^{\infty} \left(n'+1\right) a_{n'+1} x^{n'} \tag{1.64}$$

Where the last term involves making the change of variables $n = n'+1$

Substituting into equation (1.62) gives

$$\sum_{n'=0}^{\infty} \left(n'+1\right) a_{n'+1} x^{n'} = -\sum_{n=0}^{\infty} a_n x^n \tag{1.65}$$

But n' and n are dummy variables and we can compare similar powers of x by setting $n' = n$, giving the series solution

$$\sum_{n'=0}^{\infty}\left[(n+1)a_{n+1}+a_n\right]x^n=0. \qquad (1.66)$$

The above expression can be true for arbitrary x only if term by term the coefficients vanish:

$$(n+1)a_{n+1}+a_n=0. \qquad (1.67)$$

This gives rise to the recursive formula

$$a_{n+1}=-\frac{a_n}{(n+1)};\ \text{or}\ a_n=-\frac{a_{n-1}}{(n)}, \qquad (1.68)$$

with the solution

$$a_n=(-1)^n\frac{a_o}{n!}=\frac{(-1)^n}{n!}Y_0. \qquad (1.69)$$

The series solution is given by

$$Y(x)=Y_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n=Y_0e^{-x}. \qquad (1.70)$$

❖ A simple second order linear differential equation

Here is a second order differential equation for which we already know the solution:

$$Y''(x)=\frac{d^2}{dx^2}Y(x)=-Y(x). \qquad (1.71)$$

In this case, there are two linearly independent solutions and the general solution can be written as

$$Y(x) = a_0 \cos(x) + a_1 \sin(x) \tag{1.72}$$

However suppose we didn't know the solution (or at least its series expansion which amounts to the same thing.) How would go about finding two linearly independent solutions? Here symmetry comes to our help. The operator d^2/dx^2 is an even function of x, so the even and odd parts of $y(x)$ are separately solutions to equation (1.71). This suggests that we try to find series solutions of the form

$$Y(x) = \sum_{n=0}^{\infty} a_{2n+s} x^{2n+s}; \text{ for } s = 0,1 \tag{1.73}$$

If $s = 0$ we get an even function of x; and if $s = 1$, an odd function of x. Substituting this series into equation (1.71) gives

$$\begin{aligned} Y''(x) &= \sum_{n=2}^{\infty} (2n+s)(2n+s-1) a_{2n+s} x^{2n+s-2} \\ &= \sum_{n'=0}^{\infty} (2n'+s+2)(2n'+s+1) a_{2n'+s+2} x^{2n'+s} \text{ (letting } n = n'+2) \\ &= -Y = -\sum_{n=0}^{\infty} a_{2n+s} x^{2n+s} \end{aligned} \tag{1.74}$$

Comparing terms of the same power of x gives the recursion formula

$$(2n+s+2)(2n+s+1) a_{2n+s+2} = -a_{2n+s} \tag{1.75}$$

with solution

$$a_{2n+s} = (-1)^{2n} \frac{a_s}{(2n+s)!} \tag{1.76}$$

If $s = 0$, this gives a cosine series normalized to the value of a_0; and if $s = 1$, a sine series normalized to a_1, with the sum yielding the general solution given by equation (1.72).

By making the substitution $x \to mx$, we get the differential equation

$$Y''(x) = -m^2 Y(x) \tag{1.77}$$

with solutions

$$Y_m(x) = a_m \cos(mx) + b_m \sin(mx) \tag{1.78}$$

Some quick comments:

- In both of the above examples, we should have used the ratio test to find the radius of convergence of the series solutions, but we have already shown that the exponential, sine and cosine functions converge for all finite x.
- The power series expansion fails if the equation has a singularity at the expansion point. Using the Method of Forbenius in the next section, we will see how to extend the series technique to solve equations that have nonessential singularities at their origin.

1.16 Generalized power series

When a power series solution fails, one can try a generalized power series solution. This is an extension of the power series method to include a leading behavior at the origin that might include a negative or fractional power of the independent variable. A second order linear homogeneous differential equation of the form

$$y'' + f(x)\,y' + g(x)\,y = 0 \tag{1.79}$$

is said to be regular at $x = 0$ if $xf(x)$ and $x^2 g(x)$ can be written in a power series expansion about $x = 0$. That is, the singularity of $f(x)$ is not greater than x^{-1} and the singularity of $g(x)$ is not greater than x^{-2} at the origin of the expansion. Such a differential equation can be solve in terms of at least one generalized power series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s} \tag{1.80}$$

where $a_0 \neq 0$, The leading power x^s can be a negative or non integer power of x.

The statement that a_0 is the first nonvanishing term of the series, requires that terms $a_{-1}, a_{-2,\ldots}$ vanish. This constraint defines an quadratic indicial equation for s that can be solved to determine the two roots $s_{1,2}$.

❖ Fuchs's conditions

Given a regular differential equation of the form $y'' + f(x)y' + g(x)y = 0$, with solutions $s_{1,2}$ for the indicial equation:

- If $s_2 - s_1$ is non-integer, s_1 and s_2 define two linearly independent generalized power series solutions to the equation.
- If $s_2 - s_1$ is integer-valued, the two solutions may or may not be linearly independent. In the second case, the larger of the two constants is used for the first solution $y_1(x)$ and a second solution can be found by making the substitution

$$y_2(x) = y_1(x)\ln(x) + b(x) \tag{1.81}$$

  where $b(x)$ is a second generalized power series.

Example: Solve by the method of Forbenius:

$$x^2 y'' + 2xy' + x^2 y = 0. \tag{1.82}$$

Note that the differential operator is an even function of x. This suggests that we try a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n+s}, \tag{1.83}$$

where a_0 is the first nonvanishing term. Substituting (1.83) into (1.82) gives

$$\sum_{n=0}^{\infty}(2n+s)(2n+s-1)a_{2n}x^{2n+s}+\sum_{n=0}^{\infty}2(2n+s)a_{2n}x^{2n+s}$$
$$=-\sum_{n=0}^{\infty}a_{2n}x^{2n+s+2}=-\sum_{n=-1}^{\infty}a_{2n-2}x^{2n+s}, \tag{1.84}$$

which yields the recursion formula

$$\left[(2n+s)(2n+s+1)\right]a_{2n}=-a_{2n-2}. \tag{1.85}$$

Letting $n=0$ gives the indicial equation

$$\left[(s)(s+1)\right]a_0=-a_{-2}=0 \tag{1.86}$$

or

$$s=0,-1. \tag{1.87}$$

Equation (1.85) can be rewritten as

$$a_{2n}=\frac{(-1)^n}{(2n+s+1)!}a_0, \tag{1.88}$$

giving the solutions

$$y_0(x)=a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}x^{2n}=a_0\frac{\sin x}{x}, \tag{1.89}$$

$$y_{-1}(x)=a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(n)!}x^{2n-1}=a_0\frac{\cos x}{x}. \tag{1.90}$$

The general solution is given by

$$y(x)=A\frac{\sin x}{x}+B\frac{\cos x}{x} \tag{1.91}$$

In this case, the solutions can be expressed in terms of elementary functions. A solution could have been found by substituting $y = u(x)/x$ and solving for $u(x)$ to obtain $u(x) = A\sin x + B\cos x$

2. Analytic continuation

By venturing into the complex plane,
the geometric sense of multiplying by -1 can be replaced
from the operation of reflection, which is discrete,
to that of rotation, which is continuous.
The result is almost miraculous.

2.1 The Fundamental Theorem of algebra

Complex variables were introduced into algebra to solve a fundamental problem. Given a polynomial function of order N of a real variable x, how do we find its roots (zero crossings)? The equation to be solved can be written as

$$f_N(x) = \sum_{n=0}^{N} a_n x^n = 0 \tag{2.1}$$

The problem may not have a real-valued solution, it may have a unique solution, or it may have up to N distinct solutions, which are called the N roots of $f_N(x)$.

The problem can be reduced to the question of whether we can fully factor the function into N linear products . That is, does an algorithm exist that gives us uniquely

$$\begin{aligned} f_N(x) &= a_N\left(x-x_0\right)\left(x-x_1\right)\left(x-x_2\right)\ldots\left(x-x_{N-1}\right) \\ &= a_N \prod_{m=0}^{N}\left(x-x_m\right)=0 \end{aligned} \tag{2.2}$$

This problem does have a solution, but only if we allow for the possibility of complex roots. This is the *Fundamental Theorem of Algebra*, which asserts that a polynomial of order N of a complex variable z can always be completely factored into its N roots, which are complex in general

$$f_N\left(z\right)=\sum_{n=0}^{N} a_n z^n = a_N \prod_{m=0}^{N}\left(z-z_m\right). \tag{2.3}$$

❖ Conjugate pairs or roots.

If the coefficients of the power series are all real, the non-real roots always come in complex conjugate pairs. (Note that $\left(z+z_0\right)\left(z+z^*{}_0\right)=z^2+2\,\mathrm{Re}(z_0)z+\left|z_0\right|^2$ has real parameters.)

❖ Transcendental functions

If the power series is infinite, it has an infinite number of complex roots. Such functions are said to be transcendental.

2.2 The Quadratic Formula

Let's apply this to the quadratic formula given by

$$ax^2 + bx + c = a(x - x_+)(x - x_-) = 0$$
$$\text{where} \qquad (2.4)$$
$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here a, b, c are real coefficients. The roots $x_\pm$ are real if $\left(b^2 - 4ac\right) \geq 0$, and are complex if $\left(b^2 - 4ac\right) < 0$. For the later case we can rewrite the equation as

$$x_\pm = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}. \qquad (2.5)$$

❖ Definition of the square root

Consider the plot of the quadratic function $y = x^2$ shown in Figure 2-1.

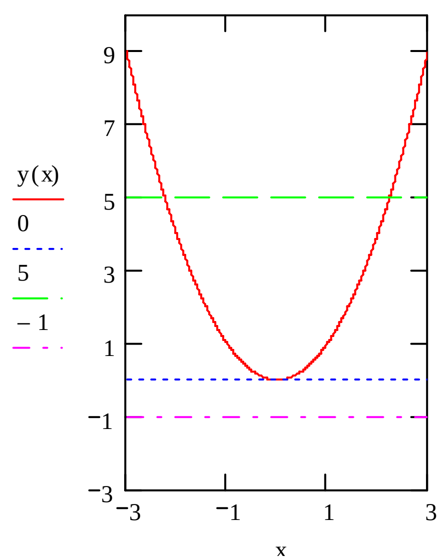


Figure 2-1 Plot of the quadratic $y = x^2$

$y = x^2$ is well defined for all x. However, the inverse function $x = y^{\frac{1}{2}}$, shown in Figure 2-1, has two real solutions for $y > 0$, one for $y = 0$, and none for $y < 0$.

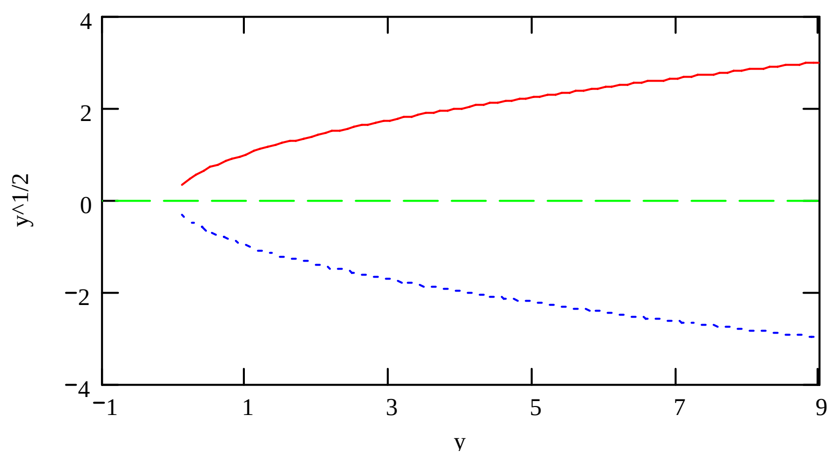


Figure 2-2 Plot of the half- root of y $y^{\frac{1}{2}} = \pm\sqrt{y}$

The *square root function* is the principal branch of $y^{\frac{1}{2}}$, which returns the positive branch of the function, i.e. $\sqrt{y} \geq 0$ for all positive y.

❖ Definition of the square root of -1

There are 2 roots of $(-1)^{1/2}$. The roots are labeled as

$$(-1)^{\frac{1}{2}} = \pm i, \tag{2.6}$$

where $i = \sqrt{-1}$ is considered to be the primary branch of the square root function. Since i is not a real number, it represents a new dimension (degree of freedom). Just as one cannot add meters and seconds, we can not add numbers on the real axis to those along the imaginary axis i .To fully understand the meaning of i , one first needs to appreciate the geometric interpretation of multiplication.

❖ The geometric interpretation of multiplication

The set of real numbers is isomorphic to a one dimensional vector function x, called the number line. Every real number x corresponds to a vector as labeled x on this line. Multiplication of the vector x by the positive number r, written as $f(x) = rx$, changes the length of the vector x by the ratio r (see Figure 2-3). Multiplication of x by $-r$: can be thought of as multiplication by r, followed by multiplication by (-1) : $f(xd) = -rx = (-1(rx))$ (see Figure 2-3) . The latter operation reflects the orientation of the vector about the origin, an improper operation. By extending the number line into a two-dimensional plane, called the complex plane, a second interpretation of multiplication by (-1) is possible, it can represent a rotation by π radians. This is important because rotations, unlike reflections, can be done continuously. The square $(-1)^2 = 1$ is understood as two rotations by π, which brings us back to starting point. $-(-r) = r$. And i can be written as the phase rotation $e^{i\pi} = -1$.

The geometric interpretation of $(\pm i)^2 = -1$ is that i is the rotation that when doubled produces a rotation by π radians. The possible answers are $e^{\pm i\pi/2} = \pm i$, where $i = e^{i\pi/2} = \sqrt{-1}$, is the principal branch of the square root function.

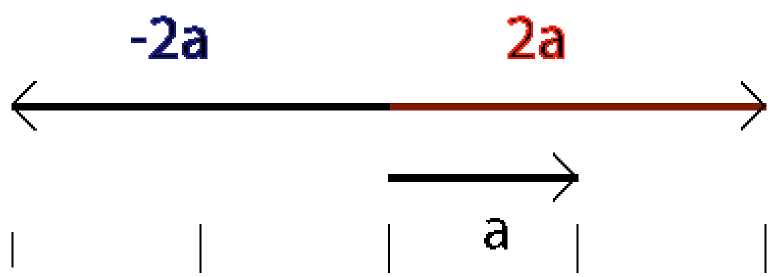


Figure 2-3 Multiplication of a point a on the real number line by a real number(± 2)

Shown in Figure 2-3 is multiplication of a vector a by(+2) and (-2). The concept of multiplication on the real number line is one of a scale change plus a possible reflection (multiplication by -1). Reflection is an improper transformation as it is discontinuous. On the complex plane this reflection is replaced by a rotation of 180°.

2.3 The complex plane

A complex number *c* can be thought of as consisting of a vector pair of real numbers (a, b) on a two dimensional plane called the complex plane. A complex vector c can be written as

$$c = a + ib. \tag{2.7}$$

Addition of complex numbers is the same as addition of 2-dimensional vectors on the plane. The plane represents all possible pairs of real numbers. Let x be an arbitrary number on the real axis, and y be a arbitrary number along the imaginary axis, then an arbitrary point on the complex plane can be referred to as

$$z = x + iy. \tag{2.8}$$

The complex plane can be quite "real" in that the properties of "real" vectors constrained to a 2-dimensional plane can be quite well represented as complex numbers in many applications. The modulus $|z|$ of a complex number z is its geometric length $|z| = \sqrt{x^2 + y^2}$. $|z|^2 = z^* z$, where z^* is the complex conjugate of z (see Figure 2-4).

Definition: *The complex conjugate of z is the complementary point on the plane given by changing the sign of i.*

$$z^* = x - iy \tag{2.9}$$

Complex Conjugate Pairs

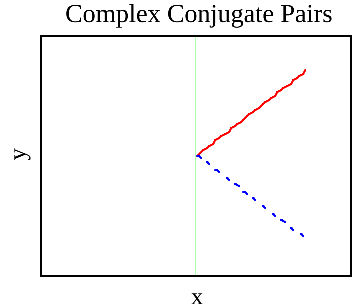


Figure 2-4 Conjugate pairs of vectors in the complex plane $x \pm iy$.

2.4 Polar coordinates

Like any 2-dimensional vector pair, (x, y), the transformation of a complex number into polar coordinates is given by the mapping

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \end{aligned} \tag{2.10}$$

using

$$z = r^{i\theta} = r\left(\cos\theta + i\sin\theta\right). \tag{2.11}$$

Therefore, a complex number can be thought of as having a real magnitude $r > 0$ and an orientation θ *wrt* (*with respect to)* the x axis. Note that the phase angle is cyclic, i.e. periodic, on interval 2π

$$e^{i(\theta+2n\pi)} = e^{i\theta}. \tag{2.12}$$

Example: Using

$$e^{ix} = \cos x + i\sin x, \tag{2.13}$$

- Derive the series expansion of $\sin x$ and $\cos x$ using the power series expansion for e^x and substituting $x \to ix$.
- Use the definition of the exponential, as the function which is its own derivative ($de^x / dx = e^x$), to prove

$$\frac{d\sin x}{dx} = \cos x,$$
$$\frac{d\cos x}{dx} = -\sin x. \tag{2.14}$$

2.5 Properties of complex numbers

Complex numbers form a division algebra, an algebra with a unique inverse for every non-zero element. Complex numbers form commutative, associative groups under both the operations of addition and multiplication. The distributive law also applies.

Definition: *Addition and subtraction of complex numbers*

Given

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2, \tag{2.15}$$

then,

$$z_3 = z_1 \pm z_2 = (x_1 + x_2) \pm i(y_1 + y_2). \tag{2.16}$$

Definition: *Multiplication of complex numbers*

$$z_3 = z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1), \tag{2.17}$$

which in polar notation becomes

$$z_3 = r_3 e^{i\theta_3} = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1 + \theta_2)}. \tag{2.18}$$

The geometric interpretation of complex multiplication is that it represents a change of scale, scaling the length by $r_3 = r_1 r_2$, and a rotation of one number by the phase of the other, with the final orientation being the sum of the two phases $\theta_1 + \theta_2$.

Definition: *Division is defined in terms of the inverse of a complex number*

$$z_1 / z_2 = z_1 \cdot z_2^{-1}$$
$$z^{-1} = \frac{z^*}{z^* z} \tag{2.19}$$
in polar notation we get
$$\left(re^{i\theta}\right)^{-1} = \frac{1}{r} e^{-i\theta}$$

Example: Calculate $\left|(2+i)/(1+i)\right|$.

$$let\ z = \frac{2+i}{1+i}$$
$$then\ |z| = \sqrt{z^* z} = \sqrt{\frac{2-i}{1-i} \cdot \frac{2+i}{1+i}} = \sqrt{\frac{4+1}{1+1}} = \sqrt{5/2}$$

Example: Calculate $z^2 = 2i$

First try it by brute force

$$z^2 = (x+iy)^2 = \left(x^2 - y^2\right) + i(2xy)$$

This leads to two real equations

$$x^2 - y^2 = 0,$$
$$2xy = 2.$$

Substituting $y = 1/x$ into the first equation, we get

$$
\begin{aligned}
&x^2-\left(\frac{1}{x}\right)^2=0,\\
&x^4-1=0,\\
&x=\mathrm{Re}\left(1^{\frac{1}{4}}\right)=\pm 1,\\
&y=\frac{1}{x}=\pm 1.
\end{aligned}
$$

Therefore

$$z=\pm(1+i).$$

Of course, the easy way is to calculate $(2i)^{\frac{1}{2}}$ directly, the way to do so will be made clear in the next section below

2.6 The roots of $z^{1/n}$

The principal root of $1^{\frac{1}{n}}=1$, since $1^n=1$ for the identity element. Using polar notation, the n^{th} distinct root of 1 is given by

$$
\begin{aligned}
&1^{\frac{1}{n}}=\left(e^{i2\pi m}\right)^{\frac{1}{n}}=e^{i2\pi m/n},\\
&\text{with distinct roots for}\\
&m=0,1...(n-1).
\end{aligned}
\tag{2.20}
$$

That is, the roots are unit vectors whose phase angles are equally spaced from the identity element 1 in steps of $2\pi/n$ as shown in Figure 2-5. This allows us to calculate the nth root of z

$$\begin{aligned}&let \quad z = re^{i\theta},\\ &then\\ &z^{\frac{1}{n}} = \left(z \cdot 1\right)^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\theta/n} e^{i2\pi m/n},\\ &for\ m = 0,1,\ldots(n-1).\end{aligned} \tag{2.21}$$

Definition: *The principal root of* $z^{1/n}$ *is defined as* $\sqrt[n]{z} = r^{1/n} e^{i\phi/n}$. *All other roots are related by uniformly spaced phase rotations of magnitude* $2\pi / n$

For example: $z^3 = 8i = 8e^{i\pi/2}$ has the solution

$$\begin{aligned}&z = \left\{2e^{i\pi/6},\ 2e^{i(\pi/6+2\pi/3)},\ 2e^{i(\pi/6+4\pi/3)}\right\},\\ &where\ e^{i\pi/6} = \cos 30^\circ + i\sin 30^\circ.\end{aligned}$$

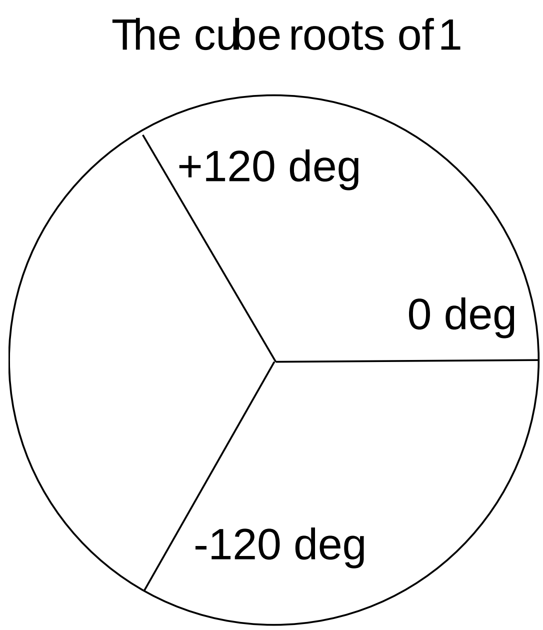


Figure 2-5 The n roots of $1^{\frac{1}{n}}$ **on the unit circle**

Shown in Figure 2-5 The n roots of $1^{\frac{1}{n}}$ on the unit circle are the cube roots of one. The concept of complex multiplication involves a phase rotation plus a change of scale. The identity element 1 is the principle root of $1^{\frac{1}{n}}$. The other $n-1$ roots are equally spaced vectors on the unit circle. The cube roots of 1 are those phase vectors that, when applied 3 times, rotate themselves into the real number 1.

2.7 Complex infinite series

A complex infinite series is the sum a real series and an imaginary series. The complex series converges if both real series and the imaginary series separately converge

$$S_c = \sum c_n = \sum (a_n + ib_n) = \sum a_n + i\sum b_n. \tag{2.22}$$

A complex series converges absolutely if the series of real numbers given by $|a_n + ib_n|$ absolutely converges.

Proof: Clearly, by the comparison test, $|a_n| \le |c_n|$ and $|b_n| \le |c_n|$, so if S_c converges absolutely, then the component series S_a and S_b converge absolutely.

The radius of convergence r of a complex power series $\sum c_n z^n$ is given by

$$r = |z| < \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|. \tag{2.23}$$

Example: Find the radius of convergence of the exponential function:

By analytic continuation e^z is found by substituting z for x in the power series representation of e^x

$$e^z = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{2.24}$$

The radius of convergence is given by

$$|z| < \lim_{n\to\infty} \left| \frac{\frac{1}{n!}}{\frac{1}{n+1!}} \right| = \lim_{n\to\infty} n+1 = \infty. \tag{2.25}$$

2.8 Derivatives of complex functions

To understand the meaning of a complex derivative, first let us remind others of the definition of the derivative for a function $f(x)$ of a real variable x. The derivative df/dx of a real valued function of x exits *iff* (*if and only if*) the limit

$$df(x)/dx = \lim_{x\to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \tag{2.26}$$

exists, and the limit is the same whether approaches zero from below or above x.

Example: The derivative of a function is undefined where the slope is undefined. Figure 2-6 shows a plot of the absolute value of x, where the derivative is undefined at the origin x=0.

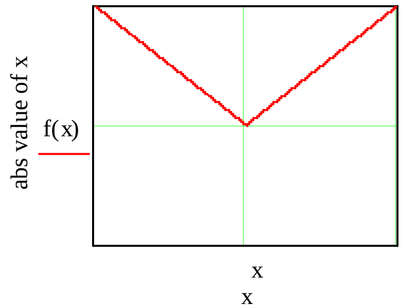


Figure 2-6 A plot of the absolute value of x.

Definition: *The derivative of a function of a complex variable at a point z_0 is given by*

$$\left.\frac{df(z)}{dz}\right|_{z=z_0} = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \tag{2.27}$$

Provided that the limit exists and is independent of the path taken by Δz in approaching z_0.

This is much more stringent condition that for the real derivative. There are an infinite number of paths that Δz can take in going to zero. This viewed as important enough so that the existence of the complex derivative is given a special name:

Definition: *A function of $f(z)$ who's derivative exists in the vicinity of a point z_0 is said to be analytic at z_0*

Note that z is an analytic function of itself:

$$\lim_{\Delta z \to 0} \frac{(z+\Delta z)-z}{\Delta z} = 1 \tag{2.28}$$

It follows (using the binomial expansion) that the derivative of z^n is also analytic:

$$\begin{aligned}
(z+\Delta z)^n &= \sum_{m=0}^{n} \binom{n}{m} z^{n-m} \Delta z^m = z^n + nz^{n-1}\Delta z + O(\Delta z^2), \\
\frac{dz^n}{dz} &= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z} = \lim_{\Delta z \to 0} \frac{nz^{n-1}\Delta z}{\Delta z} = nz^{n-1}.
\end{aligned} \tag{2.29}$$

Clearly this means that all power series in z are analytic within their radius of convergence. Note that

- Inverse powers z: z^{-n}, are singular at the origin, so are not analytic in the vicinity of $z=0$.
- Inverse power series in z, can be thought of as power series in $(1/z)$.

$$f\left(z^{-1}\right) = \sum_{n=0}^{\infty} c_n z^{-n}. \tag{2.30}$$

By the ratio test such series converge for $\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n z} \right| < 1$, or for

$$|z| > r = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|, \tag{2.31}$$

that is, they represent functions that are analytic outside of some radius of convergence.

2.9 The exponential function

The exponential function is unusual in to it has a special syntax $e(z) = e^z$; some of its most important properties are listed below.

$$\begin{aligned}
&\frac{de^z}{dz} = e^z \\
&e^{z_1}e^{z_2} = e^{(z_1+z_2)} \\
&e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad |z| < \infty \\
&e^1 = e = 2.718281828...
\end{aligned} \tag{2.32}$$

Proof: The first equation is simply the definition of the exponential function as the function that is its own derivative. The power series for e^z comes from substituting the series into the differential equation. We have already done this in the section on infinite series, just substitute $x \to z$ in the proof. The proof that $e^{z_1}e^{z_2} = e^{(z_1+z_2)}$ can be derived by substituting the series for the function in the expression, then rearranging the terms. An outline of a proof follows:

$$\begin{aligned}
e^{z_1}e^{z_2} &= \sum_{n=0}^{\infty} \frac{z_1}{n!}^{n} \sum_{m=0}^{\infty} \frac{z_2}{m!}^{m} = \left(1 + \frac{z_1}{1} + \frac{z_1^{\ 2}}{2} + ...\right)\left(1 + \frac{z_2}{1} + \frac{z_2^{\ 2}}{2} + ...\right) \\
&= \left(1 + \frac{z_1 + z_2}{1} + \frac{z_1^{\ 2} + 2z_1z_2 + z_2^2}{2} + ...\right) \\
&= \left(1 + \frac{(z_1 + z_2)}{1} + \frac{(z_1 + z_2)^2}{2} + ...\right) + ... \\
&= \sum_{n=0}^{\infty} \frac{(z_1 + z_2)^n}{n!} = e^{(z_1+z_2)}.
\end{aligned} \tag{2.33}$$

A more formal proof can be made using the binomial theorem. From that it follows that

$$\left(e^z\right)^n = \prod_{n=1}^{n} e^z = e^{nz} \tag{2.34}$$

Then, by extension, for any number c, we define

$$\left(e^z\right)^c = e^{cz} \tag{2.35}$$

These properties, given in (2.33) and (2.35), are the justification for using a power law representation for $e(z)$.

2.10 The natural logarithm

The natural logarithm is the inverse of the exponential function. Given $w = e^z$,

$$\ln(w) = z. \tag{2.36}$$

Multiplying $w_1 w_2 = e^{z_1} e^{z_2} = e^{(z_1+z_2)}$ gives

$$\ln(w_1 w_2) = z_1 + z_2 = \ln w_1 + \ln w_2. \tag{2.37}$$

$\ln(z)$ can easily be evaluated using polar notation. Let $z = re^{i\theta} = re^{i(\theta+2\pi m)}$, then

$$\ln\left(re^{i(\theta+2\pi m)}\right) = \ln\left(r\right) + \ln\left(e^{i(\theta+2\pi m)}\right). \tag{2.38}$$

Therefore, $\ln(z)$ is defined as

$$\ln z = \ln\left(r\right) + i\theta + i2\pi m \quad \text{for all m=0, }\pm 1,\ldots \tag{2.39}$$

and the of the logarithm is defined as

$$\ln z = \ln(r) + i\theta, \quad -\pi < \theta \le \pi. \tag{2.40}$$

For example,

$$\ln(i) = \ln\left(e^{i\pi/2}\right) = i\pi/2 + i2\pi m. \tag{2.41}$$

For all integer m. Therefore, the logarithm is a multivalued function of a complex variable.

2.11 The power function

The power function is defined by analytic continuation as

$$z^w = \left(e^{\ln(z)}\right)^w = e^{w\ln z}. \tag{2.42}$$

For example,

$$(i)^i = e^{i\ln i} = e^{i\ln e^{i\pi 2}} = e^{i(i\pi 2 + 2m\pi)} = e^{-(\pi/2 + 2m\pi)}. \tag{2.43}$$

Note that all the roots of $(i)^i$ are real.

This definition results in the expected behavior for products of powers:

$$z^{w_1} z^{w_w} = e^{w_1 \ln z} e^{w_2 \ln z} = e^{(w_1 + w_2)\ln z} = z^{(w_1 + w_2)} \tag{2.44}$$

2.12 The under-damped harmonic oscillator

The equation for the damped harmonic oscillator is given by

$$m\ddot{x} + b\dot{x} + kx = 0, \tag{2.45}$$

where $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$. This equation can be thought of as the projection onto the x axis of motion in a 2dimensional space given by $z = x + iy$

$$m\ddot{z} + b\dot{z} + kz = 0. \tag{2.46}$$

Let's try a solution of the form $z(t) = e^{ut}$. This leads to the quadratic equation

$$mu^2 + bu + k = 0, \tag{2.47}$$

With solutions

$$u_{\pm} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \tag{2.48}$$

If $b^2 - 4mk < 0$,

$$u_{\pm} = \frac{-b \pm i\sqrt{4mk - b^2}}{2m} \tag{2.49}$$

The solution oscillates:

$$x(t) = \left(ce^{i\omega t} + c^* e^{-i\omega t}\right)e^{-bt/2m} = \left(2\alpha \cos \omega t + 2\beta \sin \omega t\right)e^{-bt/2m}. \tag{2.50}$$

- The complex solutions are weighted sums of decaying spirals one of which rotates clockwise and the other counter-clockwise (Figure 2-7). This diagram could also represent a 2-dimensional phase space plot of position vs. momentum $p = mv$ for a 1-dimensional problem. In that case the point that they decay into is the stable point of the equations of motion ($\dot{x} = \dot{p} = 0$) which is often called the attractor.

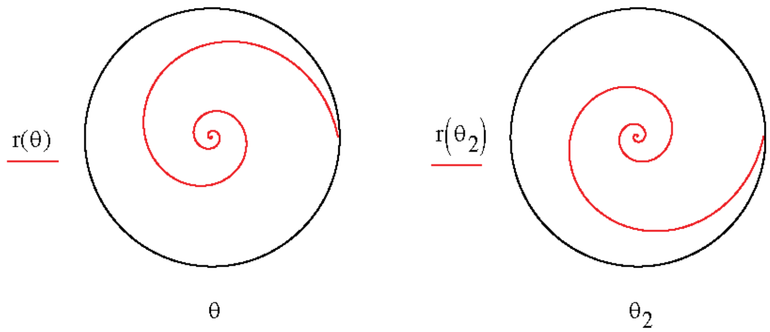


Figure 2-7 Decaying spiral solutions to the damped oscillator in the complex plane.

The total solution for z(t) is the weighted sum of the 2 complex solutions

$$z(t) = c_+ e^{u_+ t} + c_- e^{u_- t}. \tag{2.51}$$

The solution can be made real by taking the projection onto the real axis

$$x(t) = \frac{z + z^*}{2}. \tag{2.52}$$

This forces $c_\pm$ to be conjugate pairs, giving the solution

$$z(t) = \left[a\cos(\omega t) + b\sin(\omega t) \right] e^{-\lambda t}. \tag{2.53}$$

The equation of the under damped oscillator can be rewritten as

$$x(t) = A\cos(\omega t + \varphi) e^{-\lambda t}, \tag{2.54}$$

where A is the amplitude, ω is the oscillation frequency , λ is the decay rate, and φ is a phase angle that depends on the initial conditions (see Figure 2-8).The constants α and β are fixed by specifying the initial conditions

$$x(0) = x_0; \quad \text{and} \quad dx/dt(0) = v_0. \tag{2.55}$$

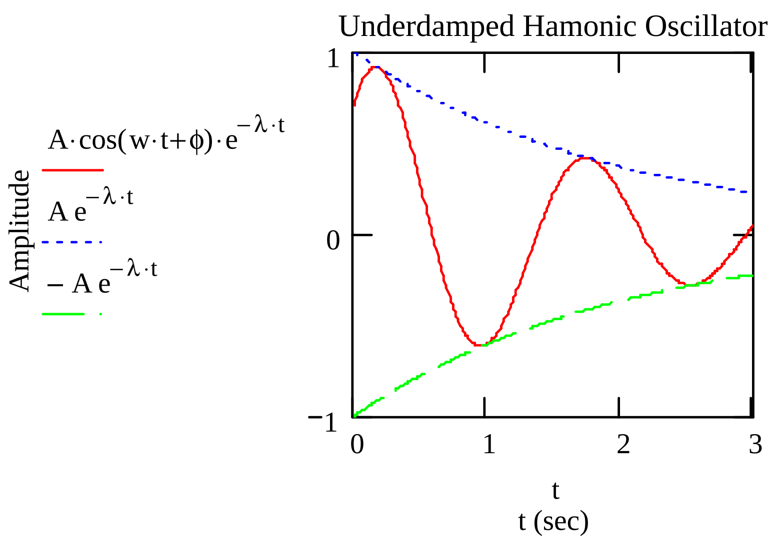


Figure 2-8 The behavior of the damped oscillator x(t), on the real axis.

2.13 Trigonometric and hyperbolic functions

All the trigonometric and hyperbolic functions are defined in terms of the exponential function e^z.

2.14 The hyperbolic functions

cosh z and sinh z are defined as the even and odd parts of the exponential function:

$$\begin{aligned} cosh(z) &= \left(\frac{e^z + e^{-z}}{2}\right) = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n!}, \\ sinh(z) &= \left(\frac{e^z - e^{-z}}{2}\right) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1!}. \end{aligned} \tag{2.56}$$

A large number of identities have been tabulated for these functions, let's look at a few

$$\begin{aligned} & cosh^2(z) - sinh^2(z) = 1, \\ & \frac{d\, cosh(z)}{dz} = sinh(z), \\ & \frac{d\, sinh(z)}{dz} = cosh(z), \\ & cosh(2z) = cosh^2(z) + sinh^2(z), \\ & sinh(2z) = 2cosh(z)sinh(z). \end{aligned} \tag{2.57}$$

The proofs all follow easily from the definitions of the functions. Some selected proofs follow:

Example: Prove $\cosh^2 z - \sinh^2 z = 1$:

$$\begin{aligned} cosh^2 z - sinh^2 z &= \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z + e^{-z}}{2}\right)^2 \\ &= \frac{e^{2z} + 2e^z e^{-z} + e^{-2z}}{4} - \frac{e^{2z} - 2e^z e^{-z} + e^{-2z}}{4} \\ &= \frac{4e^z e^{-z}}{4} = 1. \end{aligned} \tag{2.58}$$

Example: Prove that $d\sinh z/dz = \cosh z$:

$$\frac{dsinh(z)}{dz} = \frac{d}{dx}\left(\frac{e^z - e^{-z}}{2}\right) = \left(\frac{e^z + e^{-z}}{2}\right) = cosh(z). \qquad (2.59)$$

Example: Prove that $\sinh 2z = 2\sinh z \cosh z$

$$2sinh(z)cosh(z) = 2\left(\frac{e^z + e^{-z}}{2}\right)\left(\frac{e^z - e^{-z}}{2}\right) = \frac{e^{2z} - e^{-2z}}{2} = sinh(2z). (2.60)$$

2.15 The trigonometric functions

The trigonometric functions are defined as the mapping $e^z \to e^{iz}$ giving

$$\begin{aligned} e^{iz} &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \cos(z) + i\sin(z), \\ \cos(z) &= cosh(iz) = \sum_{n=0}^{\infty} \frac{(iz)^{2n}}{2n!}, \\ \sin(z) &= -isinh(iz) = -i\sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{2n+1!}. \end{aligned} \qquad (2.61)$$

Again a large number of identities have been derived for these functions, and a few of these are

$$
\begin{aligned}
&cos^2(z) + sin^2(z) = 1,\\
&\frac{dcos(z)}{dz} = -sin(z),\\
&\frac{dsin(z)}{dz} = cos(z),\\
&cos(2z) = cos^2(z) - sin^2(z),\\
&sin(2z) = 2\sin(z)\cos(z).
\end{aligned} \tag{2.62}
$$

The proofs are similar to the proofs for the hyperbolic functions. Here is an example proof, made by direct substitution:

$$
\begin{aligned}
\left[\cosh^2(iz) - \sinh^2(iz)\right] &= \left[\cos^2(z) - i^2 \sin^2(z)\right]\\
&= \left[\cos^2(z) + \sin^2(z)\right] = 1.
\end{aligned} \tag{2.63}
$$

It is reassuring to know that all the familiar trigonometric identities, that we commonly use in real analysis, carry over essentially unchanged into the complex plane.

2.16 Inverse trigonometric and hyperbolic functions

The inverse trigonometric and hyperbolic functions can be expressed in terms of the natural logarithm. However, it takes some practice to get good at this.

Example: Find $\operatorname{arcsinh}(z)$:

$$
\begin{aligned}
&w=\sinh(z)=\frac{e^{z}-e^{-z}}{2}=\frac{u+u^{-1}}{2},\\
&z=\operatorname{arcsinh}(w),\\
&u=e^{z},\\
&2wu=u^{2}-1,\\
&u^{2}-2wu-1=0,\\
&(u-w)^{2}=w^{2}+1,\\
&u=e^{z}=w\pm\sqrt{w^{2}+1},\\
&z=\ln\left(w\pm\sqrt{w^{2}+1}\right).
\end{aligned}
\tag{2.64}
$$

Solving for z

$$
\boxed{z=\sinh^{-1}(w)=\ln\left(w\pm\sqrt{1+w^{2}}\right)}
\tag{2.65}
$$

but which of the signs is correct? That depends on the problem to be solve. For example, one can find, in a book of math integrals, the following formula:

$$
\int_{0}^{x}\left(\frac{dx}{\sqrt{x^{2}-1}}\right)=\sinh^{-1}(x)>0 \text{ for } x>0,
\tag{2.66}
$$

implying that the positive branch is the correct one for the case x>0.

2.17 The Cauchy Riemann conditions

If a function $f(x,y)=u(x,y)+iv(x,y)$ is analytic in a region, the real and imaginary parts of the function satisfy the following constraints, called the Cauchy Riemann conditions:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \tag{2.67}$$

By saying that $f(x,y)$ is analytic in a region we mean that the derivative exists and is unique at each and every point in the region. The existence of the derivative implies, by the chain rule, the existence of the partial derivatives with respect to x and y. Let us consider $df\left(z\right)/dz$ calculated two different ways, first by holding y constant, then by holding x constant. In the first case we get

$$\frac{df(x,y)}{dz}=\left.\frac{\partial f}{\partial x}\right|_{y=const}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}. \tag{2.68}$$

Secondly, holding x constant gives

$$\frac{df(x,y)}{dz}=\left.\frac{\partial f}{i\partial y}\right|_{x=const}=\frac{1}{i}\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}. \tag{2.69}$$

But the two expression are the same; therefore comparing the real and imaginary parts, we get the Cauchy-Riemann conditions. The Cauchy-Riemann conditions are both necessary and sufficient conditions for a function to be analytic in a region. Basically, the proof follows from rotational invariance: these conditions have to be met on every straight line path chosen to approach the limit. Any other well behaved path can be approximated by a straight line over a small enough interval.

Example: Using the Cauchy Riemann conditions it is easy to show that $z^* = x - iy$ is not an analytic function of z, applying them we get

$$\frac{\partial x}{\partial x} \neq -\frac{\partial y}{\partial y} \tag{2.70}$$

So z^* is not an analytic function of z.

2.18 Solution to Laplace equation in two dimensions

The Laplace equation in 2dimensions can be written as

$$\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} \right) \Phi(x, y) = 0. \tag{2.71}$$

It is easy to show that, for real Φ the general solution take's the form

$$\Phi(x, y) = f(x + iy) + g(x - iy). \tag{2.72}$$

Where

$$g(z^*) = (f(z))^* \tag{2.73}$$

Proof: By direct substitution, show that $f(z)$ is a solution:

$$
\begin{aligned}
& f(x+iy) = f(z) \\
& \frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}, \quad \frac{\partial^2 f}{\partial^2 x} = \frac{d^2 f}{d^2 z}\frac{\partial z}{\partial x} = \frac{d^2 f}{d^2 z}, \\
& \frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = i\frac{df}{dz}, \quad \frac{\partial^2 f}{\partial^2 y} = i\frac{d^2 f}{d^2 z}\frac{\partial z}{\partial y} = i^2\frac{d^2 f}{d^2 z}, \\
& \therefore \quad \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} = \frac{d^2 f}{d^2 z} - \frac{d^2 f}{d^2 z} = 0.
\end{aligned}
\tag{2.74}
$$

The proof for $g(z^*)$, is similar, but, more directly, since the operator is real, if $f(z)$ is a solution, then $\left(f(z)\right)^* = g'(z^*)$ must also be a solution. If $\Phi(x, y)$ represents a real potential, then the solution takes the self-conjugate form

$$\Phi(x, y) = f(z) + \left(f(z)\right)^*. \tag{2.75}$$

3. Gamma and Beta Functions

A function that calls itself
is like a dog chasing its tail.
Where will this nonsense end?

3.1 The Gamma function

The coefficients of infinite power series are often given in terms of recursive relations. For example the series solution $e^x = \sum a_n x^n$ to the differential equation for the exponential function $de^x / dx = e^x$ leads to the following recursive formula:

$$a_n = \frac{1}{n} \cdot a_{n-1}, \tag{3.1}$$

with a solution

$$a_n = \frac{1}{n!} a_0. \tag{3.2}$$

Normalizing to $a_0 = 1$ gives

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{3.3}$$

where

$$n! = \prod_{m=1}^{n} m. \tag{3.4}$$

The factorial function occurs in the definition of Taylor's Expansion as well as in the definition of trigonometric and hyperbolic functions. This particular combinatory formula is so useful that it becomes desirable to extend its definition to non-integer values of n. The key property of the factorial is its :

$$n! = n \cdot (n-1)!. \tag{3.5}$$

It is this property that we wish to maintain as we extend it into the domain of real numbers.

❖ Extension of the Factorial function

The Gamma function represents the extension of the factorial function. Its definition must satisfy two key requirements:

$$\begin{aligned} \Gamma(p+1) &= p! && \text{for integer } p > 0, \\ \Gamma(p+1) &= p\Gamma(p) && \text{for all real } p. \end{aligned} \tag{3.6}$$

It is sufficient to define $\Gamma(p)$ in the interval $p = [1, 2]$ as recursion can be used to generate all other values. However, there exists a definite integral that has all the required properties and which is valid for all positive p. This integral is what is used to define $\Gamma(p)$ for $p > 0$. This integral is given by

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt \tag{3.7}$$

It is easy to demonstrate by integration by parts that

$$\Gamma(2) = \int_0^\infty t^1 e^{-t} dt = te^{-t}\Big|_0^\infty + \int_0^\infty e^{-t} dt = 0 - e^{-t}\Big|_0^\infty = 1 = 1! \tag{3.8}$$

And, also by integration by parts,

$$\Gamma(p+1) = \int_0^\infty t^p e^{-t} dt = t^p e^{-t}\Big|_0^\infty + p\int_0^\infty t^{p-1} e^{-t} dt = p\Gamma(p). \tag{3.9}$$

Therefore, by recursion, $\Gamma(3) = 2\Gamma(2) = 2!$, etc. The integral (3.7) meets all the necessary requirements to be the extension of the factorial function (3.4). By explicit integration, we find

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1 = 0! \tag{3.10}$$

which defines $0!$, and

$$\Gamma(0) = -1! = \int_0^\infty t^{-1} e^{-t} dt = \infty. \tag{3.11}$$

In fact, , since by using $\Gamma(p) = \Gamma(p+1)/p$

$$\begin{aligned} \Gamma(-1) &= \frac{1}{-1}\Gamma(-1+1) = -\Gamma(0) = -\infty, \\ \Gamma(-2) &= \frac{1}{-2}\cdot\Gamma(-1) = \frac{1\cdot 1}{(-2)\cdot(-1)}\cdot\Gamma(0) = \infty, \text{ et etc.} \end{aligned} \tag{3.12}$$

❖ Gamma Functions for negative values of p

Evaluation of the Gamma function for negative p is given by repeated applications of the relationship

$$\Gamma(p) = \frac{1}{p}\Gamma(p+1), \tag{3.13}$$

until $\Gamma(p+n)$ returns a positive number.

A plot of the Gamma function and its inverse is shown in Figure 3-1

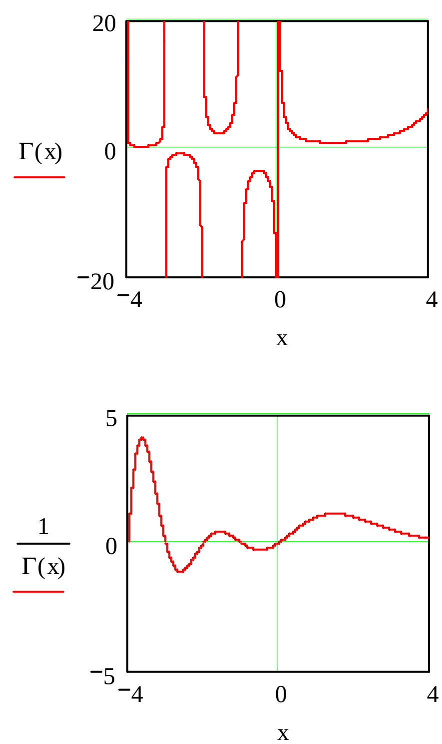


Figure 3-1 Plot of the Gamma function and its inverse

The gamma function has a shallow minimum between $1 < p < 2$ (see Figure 3-2). It blows up exponentially for large p and is divergent at p=0. For negative p, the function diverges for all negative integers. The inverse of $\Gamma(p)$, on the other hand, is quite well behaved. For positive p, it has a single maximum between $1 < p < 2$. For negative p, it oscillates, with zeros at every non-positive integer value. It is the inverse $1/n!$ that most often occurs in many series expansions,

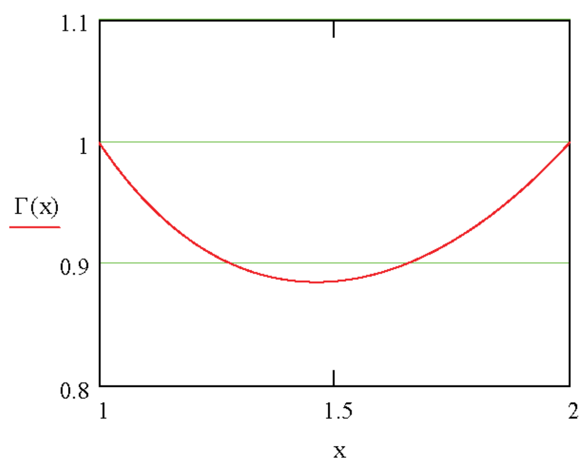


Figure 3-2 The Gamma function represents a recursive mapping of its value in the interval [1,2] of the real number line.

An important identity of the Gamma function is

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}. \tag{3.14}$$

This identity is useful in relating negative values of p to their positive counterparts. Note also that $\sin \pi p = 0$ for integer p.

Example: Show that $\Gamma(1/2)=\sqrt{\pi}$.

Use (3.14), letting $p=1/2$,

$$\begin{aligned}&\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(1-\tfrac{1}{2}\right)=\Gamma^2\left(\tfrac{1}{2}\right)=\frac{\pi}{\sin\left(\pi/2\right)}=\pi,\\&\therefore\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}.\end{aligned}\tag{3.15}$$

Example: Find $\Gamma(-3/2)$.

Use the recursive property of the gamma function:

$$\begin{aligned}&\Gamma\left(p\right)=\Gamma\left(p+1\right)/p,\\&\Gamma\left(-\tfrac{3}{2}\right)=\frac{1}{-\frac{3}{2}}\Gamma\left(-\tfrac{1}{2}\right)=\left(\frac{1}{-\frac{3}{2}}\right)\left(\frac{1}{-\frac{1}{2}}\right)\Gamma\left(\tfrac{1}{2}\right)=\frac{4}{3}\sqrt{\pi}.\end{aligned}\tag{3.16}$$

❖ Evaluation of definite integrals

An important use of Gamma functions is in the evaluation of definite integrals. In fact, its definition for positive p can be thought of as defining the normalization of a family of integrals. By making a change of coordinates, the normalization of many other useful integrals can be found.

Example: Transformation of coordinates $t=x^2$.

$$\begin{aligned}&t=x^2\Rightarrow x=\sqrt{t};\quad dt=2xdx;\\&\Gamma(p)=\int_0^\infty t^{p-1}e^{-t}dt=2\int_0^\infty x^{2(p-1)}e^{-x^2}xdx.\end{aligned}\tag{3.17}$$

Therefore,

$$\Gamma(p) = 2\int_0^\infty x^{2p-1} e^{-x^2}\, dx. \tag{3.18}$$

Example: Find the normalization of a the Normal Gaussian Distribution

The Normal Distribution of a statistical measurement of a quantity X, centered at a mean X_0 and having a random rms error spread σ, is given by

$$y(x) = Ne^{-x^2/2}, \tag{3.19}$$

where $x = (X - X_0)/\sigma$. This distribution is normalized such that

$$\int_{-\infty}^{\infty} y(x)dx = N\int_{-\infty}^{\infty} e^{-x^2/2} dx = 1. \tag{3.20}$$

Since the integrand is symmetric we can rewrite this as

$$\int_0^\infty e^{-x^2/2} dx = \frac{1}{2N}. \tag{3.21}$$

Let's make the change of variable

$$\begin{aligned} x' &= x^2/2 \Rightarrow x = \sqrt{2x'}, \\ dx' &= xdx \Rightarrow dx = \frac{1}{\sqrt{2}} x'^{-1/2} dx', \end{aligned} \tag{3.22}$$

Then, by substitution,

$$\int_0^\infty e^{-x^2/2}dx = \frac{1}{\sqrt{2}}\int_0^\infty x'^{-1/2}e^{-x'}dx' = \frac{1}{\sqrt{2}}\Gamma\left(\tfrac{1}{2}\right) = \frac{1}{2N}, \tag{3.23}$$

which leads to the following formula

$$N = \frac{\sqrt{2}}{2}\frac{1}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{2\pi}}. \tag{3.24}$$

Therefore, the "normalized" is given by

$$y(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}; \quad \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = 1. \tag{3.25}$$

3.2 The Beta Function

Another statistical combination that often reoccurs is represented by the binomial coefficients, given by

$$\binom{n}{m} = \frac{n!}{(n-m)!m!}. \tag{3.26}$$

In probability theory, they denote the number of ways one can arrange n objects taken m at a time. These coefficients have already been seen in the Binomial formula for integer powers of n,

$$(A+B)^n = \sum_{m=0}^{n}\binom{n}{m}A^{n-m}B^m. \tag{3.27}$$

Again, one would like to extend this formula to the real number domain. This is done by defining the B (Greek capital beta) function given by

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{3.28}$$

Clearly, the Beta function is symmetric under interchange of indices

$$\mathrm{B}(p,q) = \mathrm{B}(q,p), \tag{3.29}$$

and, for integer values of $p = n$ and $q = m$,

$$\mathrm{B}(n+1,m+1) = \frac{n!(m)!}{(n+m+1)!} = \frac{1}{n+m+1}\frac{n!(m)!}{(n+m)!} \tag{3.30}$$

$$\binom{n+m}{m} = \frac{1}{(n+m+1)\mathrm{B}(n+1,m+1)}. \tag{3.31}$$

The Beta function is useful in the determining normalization of many common integrals. Among them are the canonical forms

$$\begin{aligned}
&\text{for } p>0,\, q>0: \\
&\mathrm{B}(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}\, d\theta, \\
&\mathrm{B}(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1}\, dx, \\
&\mathrm{B}(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\, dx.
\end{aligned} \tag{3.32}$$

These three integral forms are related to one another by the transformations

$$x = \sin^2\theta = y/(1+y). \tag{3.33}$$

A number of other definite integrals that can be put into one of these forms.

The following proof that the above functions are indeed equivalent to the defining equation for the Beta function makes use of the fact that the integration over the surface of a quadrant $0 < x < \infty, 0 < y < \infty$ is integration over a quarter-circle. When we change to polar coordinates the range of angles is $0 < \theta < \pi/2$, giving

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty x^{2p-1}e^{-x^2}dx\int_0^\infty y^{2q-1}e^{-y^2}dy. \tag{3.34}$$

$$\begin{aligned}&\mathit{Letting}\left(x = r\cos\theta, y = r\sin\theta, dxdy = rdrd\theta\right):\\ &= 4\int_0^{\pi/2}\left(\cos\theta\right)^{2p-1}\left(\sin\theta\right)^{2q-1}d\theta\int_0^\infty r^{2(p+q)-1}e^{-r^2}dr,\end{aligned} \tag{3.35}$$

and therefore,

$$\Gamma\left(p\right)\Gamma\left(q\right) = 2\Gamma(p+q)\int_0^\pi\left(\cos\theta\right)^{2p-1}\left(\sin\theta\right)^{2q-1}d\theta = \Gamma(p+q)B(p,q). \tag{3.36}$$

3.3 The Error Function

The Error function (Figure 3-3) can be considered as an incomplete integral over a gamma function or “”. are defined as the partial integrals of the form

$$\gamma(t, p) = \int_0^t t^{p-1}e^{-t}dt. \tag{3.37}$$

Often, it is desirable to use the normalized incomplete gamma functions instead

$$\widetilde{\gamma}(t,p)=\frac{\int_0^t t^{p-1}e^{-t}dt}{\Gamma(p)}. \tag{3.38}$$

For example, the normalization of the Gaussian integral is given by

$$\Gamma(1/2)=2\int_0^\infty e^{-x^2}dx. \tag{3.39}$$

and the error function is defined as the normalized incomplete integral

$$Erf(x)=\frac{2\int_0^x e^{-x^2}dx}{\Gamma(1/2)}=\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx. \tag{3.40}$$

This integral can be expanded in a power series, giving

$$Erf(x)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}x^{2n+1}, \tag{3.41}$$

which is useful for small x.

The is defined as

$$erfc(x)=1-erf(x). \tag{3.42}$$

The Error function is closely related to the likelihood of error in a measurement of normally distributed data. However, like many standard mathematical functions, the normalization is slightly different from how physicists would like to see it defined. Its relation to the Gaussian probability distribution is given by

$$P(-x,x)=\frac{1}{\sqrt{2\pi}}\int_{-x}^{x}e^{-x'^2/2}dx'=erf\left(\frac{x}{\sqrt{2}}\right). \tag{3.43}$$

This returns the probability that a measurement of normally distributed data falls in the interval $[-x,x]$. Books of mathematical tables will tabulate at least one of these two functions, if not both.

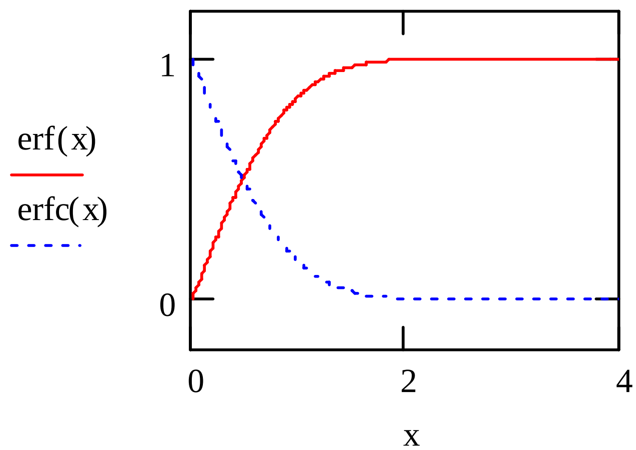


Figure 3-3 The Error function and the complementary Error function

3.4 Asymptotic Series

Figure 3-3 shows that the error function converges to its sum rapidly. Indeed the challenge is to measure error probabilities that are small but non-zero at large *x*. For example, there is a major difference between saying "a proton never decays" and that of saying “a proton rarely decays”. Since the universe is still around, the probability must be very, very small; but if we want to quantify this probability, then we must be able to calculate small deviations from zero.

Taylor’s expansion in terms of a convergent power series works well for small x. But, at large x, it is convenient to expand the function in inverse powers of x. However, in this case it turns out that that expansion doesn’t converge. For problems like these, the concept of an asymptotic series was created.

Let’s look at how we might calculate the complementary function from first principles. Define

$$erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt. \tag{3.44}$$

We can try to solve this by integrating by parts by making the substitutions

$$e^{-t^2} = \frac{te^{-t^2}}{t}, \quad e^{-t^2} dt = \frac{1}{2t} e^{-t^2} dt^2 = \frac{-1}{2t} d\left(e^{-t^2}\right), \tag{3.45}$$

in

$$\begin{aligned} \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt &= \frac{2}{\sqrt{\pi}} \left(\frac{-1}{2t} e^{-t^2} \Big|_x^{\infty} + \int_x^{\infty} \frac{1}{2t^2} e^{-t^2} dt \right) \\ &= \frac{2}{\sqrt{\pi}} \left(\frac{1}{2x} e^{-x^2} + \int_x^{\infty} \frac{1}{2t^2} e^{-t^2} dt \right), \end{aligned} \tag{3.46}$$

where

$$\int_x^\infty \frac{1}{2t^2}e^{-t^2}dt < \frac{1}{2x^2}\int_x^\infty e^{-t^2}dt. \tag{3.47}$$

Therefore, after the first integration by parts, the fractional error in the remainder is less than $\frac{1}{2x^2}$. This is a small error if x is large enough. We can repeat the process if we are not satisfied, getting the asymptotic series

$$erfc(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1-\frac{1}{2x^2}+\frac{1\cdot 3}{\left(2x^2\right)^2}-\frac{1\cdot 3\cdot 5}{\left(2x^2\right)^3}+\cdots\right). \tag{3.48}$$

For a number of iterations, integration by parts improves the error, but, after a while, the error begins to grow again (there is a double factorial hiding in the numerator). Therefore, there are an optimum number of integration by parts to make. The series, with the partial sum S_N taken to infinity, does not converge. Nevertheless the error in the finite series (for fixed N) goes to zero as $x \to \infty$. This is the difference between the definition of a convergent series and an asymptotic series. A convergent series is convergent for a given x, i.e., holding x constant, one takes the limit $N \to \infty$; The asymptotic series holds N constant and takes the limit $x \to \infty$.

Definition: *A series f(x) is said to be an asymptotic series in* x^{-n}, *written as*

$$f(x) \sim \sum_{n=0}^{\infty}\frac{a_n}{x^n}, \tag{3.49}$$

if the absolute value of the difference of the function and the partial sum goes to zero faster than x^{-N} for fixed N as $x \to \infty$

$$\lim_{x\to\infty}\left| f(x) - \sum_{n=0}^{N} \frac{a_n}{x^n} \right| \cdot x^N \to 0. \tag{3.50}$$

It is possible for as series to be both convergent and asymptotic—e.g., all power convergent series in x can be said to be asymptotic as $(x \to 0)$—but the non-convergent case is the most interesting one.

Asymptotic series often occur in the solution of integrals of the following kind:

$$I_1(x) = \int_x^{\infty} e^{-u} f(u) du \tag{3.51}$$

Or of the type

$$I_2(x) = \int_x^{\infty} e^{-u} f\left(\frac{u}{x}\right) du, \tag{3.52}$$

where $f(u/x)$ is expanded in a power series in u/x.

❖ Sterling's formula

is a good example of a non-convergent asymptotic series. It is given by

$$\Gamma(p+1) \sim \sqrt{2\pi x} \cdot x^x \cdot e^{-x} \left\{ 1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} + \cdots \right\}. \tag{3.53}$$

If p is as small as 10, stopping after the second term gives an error on the order of 50 ppm. For very large p,

$$p! \sim \sqrt{2\pi p} \cdot p^p e^{-p} \tag{3.54}$$

is a good approximation to the factorial function, where the $\sim$ indicates that the ratio of the two sides approaches 1 as $p \to \infty$.

Discussion Problem: The Exponential Integral

The integral

$$Ei(x) = \int_x^\infty \frac{e^{-t}}{t} dt \tag{3.55}$$

is called the Exponential Integral. Note that it diverges as $x \to 0$.

- Find the asymptotic expansion for the exponential integral.
- Express $\int_0^x 1/\ln(1/t)dt$ as an exponential integral.

4. Elliptic Integrals

When Kepler replaced the epicycles
of the ancients with ellipses,
he was onto something special.

Books of integral tables tabulate and catalog integrals in terms of families with a certain generic behavior. For example, a large number of integrals can be categorized as a rational function of x times a radical of the form $\sqrt{ax^2+bx+c}$ The solution of integrals of this general form almost always can be expressed in terms of elementary trigonometric functions or hyperbolic functions. For example, substituting $x=\sin\theta$,

$$\begin{aligned}\int\sqrt{1-x^2}\,dx &= \int\cos^2\theta\, d\theta = \frac{\theta}{2}+\frac{\sin 2\theta}{4}+C \\ &= \tfrac{1}{2}\left(\arcsin(x)+x\sqrt{1-x^2}\right)+C.\end{aligned} \tag{4.1}$$

Or a similar example:

$$\int\frac{1}{\sqrt{1-x^2}}dx=\theta=\arcsin x. \tag{4.2}$$

Elliptic functions were introduced to allow the solutions to the large class of problems of the form

$$\int R(x,y)dx \tag{4.3}$$

where $R(x,y)$ is any rational function of x and y, and

$$y(x) = \sqrt{ax^4 + bx^3 + cx^2 + dx + e}. \tag{4.4}$$

The more complete math tables will have many pages of examples of integrals of this type, solved in terms of one of three standard forms of elliptic integrals, called the elliptic integrals of the 1st, 2nd, and 3rd kinds. We will look at some detail at the first 2 kinds of elliptic integral. The elliptic integral of the 3rd kind is less frequently seen in elementary physics texts. These integrals are usually expressed in one of two standard forms, called the form and the forms of the integrals. The Elliptic integral of the 2nd kind is related to the arc length of an ellipse, which lent its name to this class of integrals. Therefore we will examine the integrals of the second kind first.

4.1 Elliptic integral of the second kind

The Jacobi form for the incomplete elliptic integral of the 2nd kind is given by

$$E(k,x) = \int_0^x \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx \quad \text{for } 0 \le k \le 1. \tag{4.5}$$

Note that $\left(1-k^2x^2\right)\left(1-x^2\right)$ has 4 real roots at $x = \pm 1, \pm 1/k$.

Letting $x = \sin\phi$, the Legendre form of the integral is given by

$$E(k,\phi)=\int_0^{\phi}\sqrt{1-k^2\sin^2\phi}\,d\phi \quad \text{for } 0\le k\le 1. \tag{4.6}$$

A plot of the Legendre form, shown in Figure 4-1, is illustrative of the general behavior. The integrand is periodic on interval $[0,\pi]$, so the functions are sometimes said to be "doubly-periodic" in ϕ. The integral is a repeating sum of the form

$$E(k,\phi+n\pi)=2nE(k)+E(k,\phi) \tag{4.7}$$

where $E(k)$ is the given by

$$E(k)=E(k,\pi/2)=\int_0^1\frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\phi}\,d\phi\ . \tag{4.8}$$

By symmetry about $\pi/2$, it is only necessary to tabulate the integral from $[0,\pi/2]$.

$$\begin{gathered}E(k,\phi+\pi/2)=2E(k)-E(k,\phi),\\ (0\le\phi\le\pi/2).\end{gathered} \tag{4.9}$$

The integral over an even integrand is an odd function so

$$E(k,\phi)=-E(k,\phi). \tag{4.10}$$

Combining the above results gives

$$E(k,n\pi\pm\phi)=2nE(k)\pm E(k,\phi) \text{ for } 0\le\phi\le\pi/2. \tag{4.11}$$

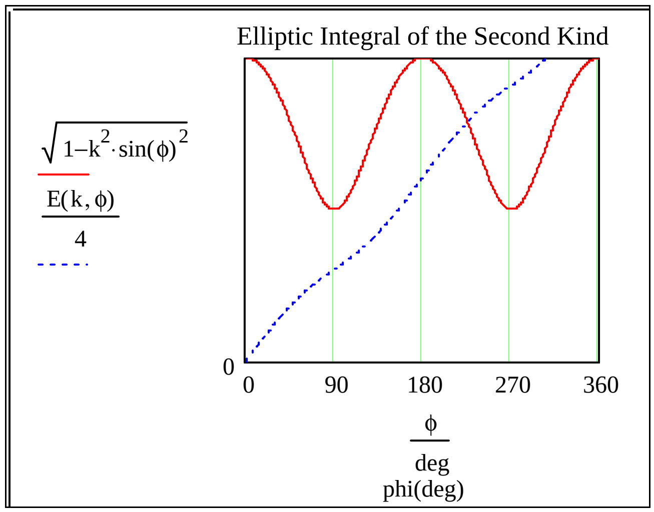


Figure 4-1 Elliptic Integral of the second kind (k=sin(60 deg))

Example: Calculate the arc length of a segment of an ellipse.

An ellipse has a semi-major axis of length a and a semi-minor axis of length b (see Figure 4-2). Aligning the ellipse with the semi-minor axis along the x-axis, it can be described by the following two parametric equations:

$$x = b\cos\phi,$$
$$y = a\sin\phi. \tag{4.12}$$

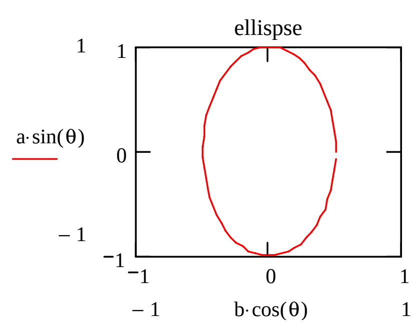


Figure 4-2 diagram of an ellipse with $a = 1, b = 0.5$

An element of arc-length can be written as

$$
\begin{aligned}
ds &= \sqrt{d^2x + d^2y} = \sqrt{b^2\sin^2\phi + a^2\cos^2\phi}\,d\phi \\
&= a\sqrt{1 - \frac{a^2 - b^2}{a^2}\sin\phi}\,d\phi = a\sqrt{1 - k^2\sin\phi}\,d\phi,
\end{aligned} \tag{4.13}
$$

where $k = \sqrt{1 - (b/a)^2} = e$ is the eccentricity of the ellipse; $k = 0$ is a circle; and $k = 1$ is a vertical line.

Integrating along ϕ gives

$$
\int_0^\theta a\sqrt{1 - k^2\sin^2\phi}\,d\phi = aE(e,\phi). \tag{4.14}
$$

The circumference of the ellipse is found by integrating over a complete revolution

$$
C = aE(k, 2\pi) = 4aE(k). \tag{4.15}
$$

Verify:

$$
\begin{aligned}
k &= 0, && \text{circle,} && C = 4aE(0) = 2\pi a, \\
k &= 1, && \text{straight line,} && C = 4aE(1) = 4a.
\end{aligned} \tag{4.16}
$$

Since the orbits of planets are ellipses, $E(k,\phi)$ is a very valuable function.

There are several ways common ways of calculating $E(k,\phi)$:

- Look up the tabulated value in a book of integral tables (the common way, before the invention of personal computers).
- Use a high level math program like Maple or Mathematica.

- Use a scientific programming library, and your favorite programming language.
- Expand the integral in a power series in $\sin^2\theta$ (converges rapidly for small k).

4.2 Elliptic Integral of the first kind

The elliptic integral of the first kind occurs in the solution of many classical mechanics problems, including the famous one of the simple pendulum. Whole books have be written about it. The Jacobi form of the integral is given by

$$F(k,x)=\int_0^x \frac{1}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}dx \quad \text{for } 0\le k\le 1. \tag{4.17}$$

And, letting $x=\sin\phi$, the Legendre form is given by

$$F\left(k,\phi\right)=\int_0^\theta \sqrt{1-k^2\sin^2\phi}d\phi \quad \text{for } 0\le k\le 1. \tag{4.18}$$

The is given by

$$K(k)=F(k,\pi/2)=\int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\phi}}d\phi\ . \tag{4.19}$$

Figure 4-3 shows a plot of the elliptic integral of the 1st kind, for $k=\sin 60^\circ$. The same kind of symmetry arguments used in discussing the Integrals of the 2nd kind apply here, F is doubly periodic in ϕ, and, by symmetry, only the values between $[0,\pi/2]$ need to be tabulated. In general

$$F(k,n\pi\pm\phi)=2nK(k)\pm F(k,\phi)\ \text{for}\ 0\le\phi\le\pi/2. \tag{4.20}$$

Table 4-1 tabulates the values of the complete elliptic integrals of the first and second kind. When $k=0$, one has a circle and the value of the integral is $\pi/2$. For $k=1$, the complete elliptic integral of the 1st kind diverges.

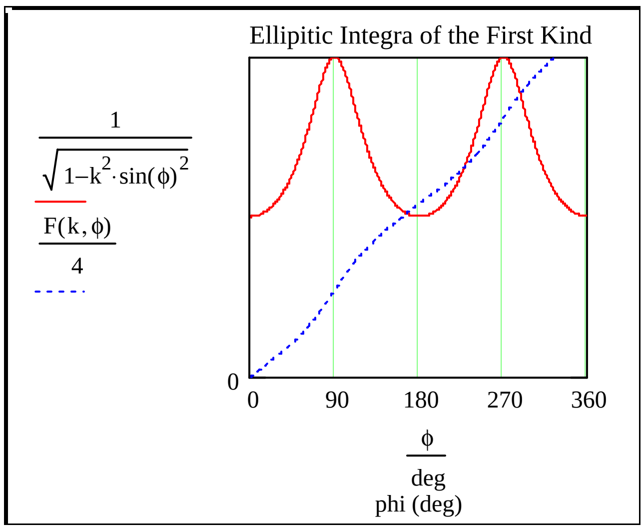


Figure 4-3 Elliptic Integral of the first kind (k=sin(60 deg))

Integrands of Elliptic Integrals are periodic on interval 180 deg and are symmetric about half that interval; therefore, they are generally only tabulated in the interval [0, 90] deg.

Table 4-1 Complete Elliptic Integrals of the first and second kind

Complete Elliptic Integrals of the 1st and 2nd kind

ψ	$E(\sin(\psi))$	$K(\sin(\psi))$
0	1.571	1.571
5	1.568	1.574
10	1.559	1.583
15	1.544	1.598
20	1.524	1.62
25	1.498	1.649
30	1.467	1.686
35	1.432	1.731
40	1.393	1.787
45	1.351	1.854
50	1.306	1.936
55	1.259	2.035
60	1.211	2.157
65	1.164	2.309
70	1.118	2.505

75	1.076	2.768
80	1.04	3.153
85	1.013	3.832
90	1	∞

Example: By substituting $\sin(\theta/2) = k\sin\phi = \sin(\theta_{\max}/2)\sin\phi$, and using $\cos\theta = 1 - 2\sin^2(\theta/2)$, show that $F(k,\phi)$ can be written as

$$F(k,\phi) = \frac{1}{\sqrt{2}}\int_0^\theta \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_{\max}}}. \tag{4.21}$$

Proof: Work the problem backwards from the result: Begin by calculating the change in derivatives:

$$\begin{gathered} d\sin(\theta/2) = \frac{1}{2}\cos(\theta/2)\,d\theta = kd\sin\phi = k\cos\phi d\phi, \\ d\theta = \frac{2k\cos\phi}{\cos(\theta/2)}d\phi. \end{gathered} \tag{4.22}$$

Next, make the necessary substitutions into the integral:

$$
\begin{aligned}
F &= \frac{1}{\sqrt{2}}\int_0^\theta \frac{d\theta}{\sqrt{\left(1-2\sin^2(\theta/2)\right)-\left(1-2\sin^2(\theta_{\max}/2)\right)}},\\
&= \frac{1}{2}\int_0^\theta \frac{d\theta}{\sqrt{\sin^2(\theta_{\max}/2)-\sin^2(\theta/2)}} = \frac{1}{2}\int_0^\theta \frac{d\theta}{\sqrt{k^2-\sin^2(\theta/2)}},\\
&= \frac{1}{2}\int_0^\theta \frac{d\theta}{\sqrt{k^2-k^2\sin^2\phi}} = \frac{1}{2k}\int_0^\theta \frac{d\theta}{\cos\phi} = \frac{1}{2k}\int_0^\phi \frac{d\phi}{\cos\phi}\frac{2k\cos\phi}{\cos(\theta/2)},\\
&= \int_0^\phi \frac{d\phi}{\cos(\theta/2)} = \int_0^\theta \frac{d\phi}{\sqrt{1-\sin^2\theta/2}} = \int_0^\theta \frac{d\phi}{\sqrt{1-k^2\sin^2\varphi}}.
\end{aligned}
\tag{4.23}
$$

4.3 Jacobi Elliptic functions

A rich literature has grown around the topic of the elliptic integral of the 1st kind, with a specialized language and names for functions. To see where this language comes from consider the simpler circular integral which can be obtained by letting $k=0$ in the Jacobi form:

$$
\begin{aligned}
&u = F(0,x) = \int_0^x \frac{1}{\sqrt{1-x^2}}dx = \phi = \arcsin x,\\
&sn(u) = x = \sin\phi.
\end{aligned}
\tag{4.24}
$$

Now let's generalize this terminology for $k \neq 0$, in which case $u \neq \phi$. We call ϕ the amplitude of u

$$amp(u) = \phi. \tag{4.25}$$

and denote the inverse function F^{-1} by the special name so that

$$\begin{aligned} u = sn^{-1}x = F(k,\phi) &= \int_0^x \frac{1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}dx \\ &= \int_0^\phi \frac{1}{\sqrt{1-k^2\sin^2\phi}}d\phi, \\ sn(u) = x = \sin\phi &= \sin(amp(u)). \end{aligned} \tag{4.26}$$

sn is pronounced roughly as "ess-en". (Try saying three times fast: "ess-en u is the sine of the amplitude of u"). A plot of $sn(u)$ vs. u looks very similar to a sine wave as seen in Figure 4-4.

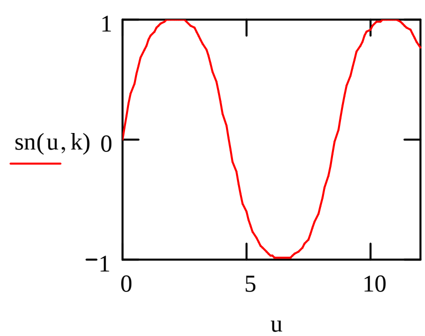


Figure 4-4 A plot of $sn(u) = x = \sin\phi$

Just as a number of trigonometric identities have been developed over the years, the same is true for the elliptic functions. Several of the more basic relationships are given by:

$$cn(u) = \cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-\mathrm{sn}^2(u)} = \sqrt{1-x^2}, \tag{4.27}$$

$$dn(u) = \frac{d\phi}{du} = \left(\frac{du}{d\phi}\right)^{-1} = \sqrt{1-k^2\sin^2\phi} = \sqrt{1-k^2\,\mathrm{sn}^2\,u}, \tag{4.28}$$

$$\frac{d}{du}sn(u) = \frac{d}{du}\sin\phi = \cos\phi\frac{d\phi}{du} = cn(u)dn(u). \tag{4.29}$$

Example: The simple pendulum

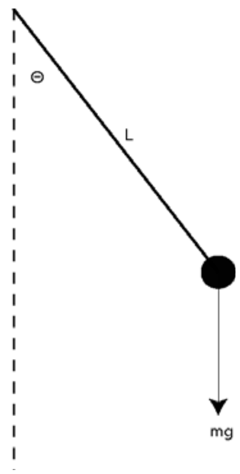


Figure 4-5 Simple Pendulum

The simple pendulum (Figure 4-5) satisfies the conservation of energy equation

$$\frac{I\dot{\theta}^2}{2}+mgl\left(\cos\theta_{\max}-\cos\theta\right)=0, \tag{4.30}$$

where $I=ml^2$. Solving,

$$\frac{1}{\sqrt{2}}\int_0^{\theta}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_{\max}}}=\int_0^1\sqrt{g/l}\,dt, \tag{4.31}$$

$$\begin{aligned}
\omega_0 t &= \frac{1}{\sqrt{2}}\int_0^{\theta}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_{\max}}} = F(k,\varphi) = u, \\
\sin\theta/2 &= k\sin\varphi, \\
k &= \sin\theta_{\max}/2, \\
x &= \frac{\sin\theta/2}{\sin\theta_{\max}/2} = sn(u) = sn\left(\omega_0 t,k\right), \\
\sin\theta/2 &= k\bullet sn\left(\omega_0 t,k\right), \\
\theta &= 2\arcsin\left(k\bullet sn\left(\omega_0 t,k\right)\right).
\end{aligned} \tag{4.32}$$

The depends on its amplitude and is given by

$$\omega_0 T = F\left(k=\sin\left(\theta_{\max}/2\right),2\pi\right) = 4K\left(\sin\left(\theta_{\max}/2\right)\right). \tag{4.33}$$

Let's put in some numbers, choosing

$$\begin{aligned}
&m = 1\text{ kg}, \\
&L = 1\text{ m}, \\
&\omega_0 = \sqrt{g/l} = \sqrt{9.8/\mathrm{s}^2} = 3.13\text{ rad/s}, \\
&\theta_{\max} = 60^{\circ}, \\
&k = \sin\theta_{\max}/2 = \sin 30^{\circ} = 1/2.
\end{aligned} \tag{4.34}$$

Then, the period of the pendulum is

$$\begin{aligned}
\omega_0 T &= F\left(k=1/2,2\pi\right), \\
T &= \frac{1}{\omega_0}4K\left(1/2\right), \\
T &= \frac{1}{3.13}4\left(1.686\right) = 2.15\text{ s},
\end{aligned} \tag{4.35}$$

Compare this to the small amplitude limit

$$T_0\big|_{smallOsc} = \frac{2\pi}{\omega_0} = 2.01\text{ s}. \tag{4.36}$$

Note the 7% difference from the small oscillation behavior. Never use a simple pendulum to tell time! The analytic solution to the simple pendulum for the conditions studied is shown in Figure 4-6.

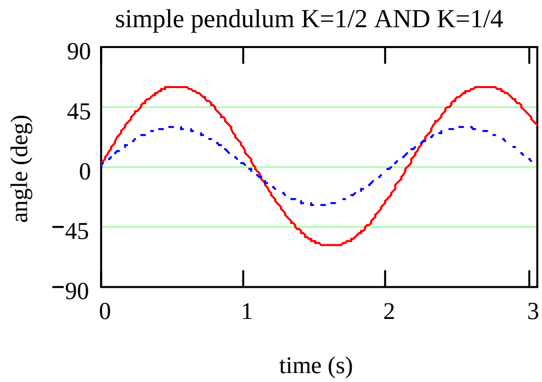


Figure 4-6 Plot of angle vs. time for the simple pendulum. Note that the zero crossing time of the period depends on the amplitude.

4.4 Elliptic integral of the third kind

For completeness, here is the definition of the Elliptic Integral of the 3rd kind: The Legendre form of the Incomplete Integral is

$$\Pi(k,n,\phi)=\int_0^{\phi}\frac{d\phi}{\left(1+n\sin^2\phi\right)\sqrt{1-k^2\sin^2\phi}}. \tag{4.37}$$

And, the Complete Integral is given by

$$\Pi(k,n,\pi/2)=\int_0^{\pi/2}\frac{d\phi}{\left(1+n\sin^2\phi\right)\sqrt{1-k^2\sin^2\phi}}. \tag{4.38}$$

5. Fourier Series

Time is measured in cycles.
The earth rotates around the sun.
Atoms oscillate. Patterns repeat.

5.1 Plucking a string

What happens when one plucks a string on a stringed instrument? The fundamental harmonic is given by the length of the string, its mass density and its tension. Depending on where we pluck the string, one can choose to emphasize different harmonics. After this point, it starts to get complicated, as the shape and nature of the sound board will further modify the sound. This problem will be analyzed in some detail when we study the wave equation. But in general, if one does a Fourier decomposition of the wave form, only multiples of the fundamental frequency will contribute. In the case of a plucked string, the boundary conditions are due to the clamping of the string, which removes all other frequencies. Is this effect real, or is it a mathematical connivance? It is definitely real, one can hear it. The human ear is a pretty good frequency analyzer. In this section we will explore how to decompose a periodic function into its Fourier series components. These components are a solution to an eigenvalue problem. The eigenfunctions represent the possible normal modes of oscillation of a periodic function. In the case of the plucked string, the motion is periodic in time. In other cases, we might be dealing with a cyclic variable, say the rotation angle of a planet as it makes its path around its sun. We begin with the simplest of models: a one dimensional rotation angle. If one defines a field on a circle, consistency requires that a rotation by 2π must give the same field.

5.2 The solution to a simple eigenvalue equation

In solutions to partial differential equations in cylindrical or spherical coordinates, the technique of separation of variables often leads to the following very simple equation for the azimulthal coordinate ϕ

$$\frac{d^2 f(\phi)}{d\phi^2} = -m^2 f(\phi) \tag{5.1}$$

where $f(\phi)$ satisfies periodic boundary conditions

$$f(\phi + 2\pi) = f(\phi) \tag{5.2}$$

The solutions of this equation are very well known —think "simple harmonic oscillator"— The exponential form of the solution is

$$f_m(\phi) = c_m e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \cdots \tag{5.3}$$

The requirement that $f_m(\phi)$ be periodic on interval 2π restricts the eigenvalues to positive and negative integers. The case $m=0$ is a special case in that $m=\pm 0$ represents the same eigenvalue.

Often it is convenient to solve the equation in terms of sine and cosine functions. Using $e^{\pm im\phi}=\cos m\phi \pm i\sin m\phi$, we find the real solutions to the eigenvalue equation to be

$$f_m(\phi)=\begin{cases}\dfrac{a_0}{2} & \text{m=0}\\ a_m\cos(m\phi)+b_m\sin(m\phi) & \text{m=1,2,3,}\ldots\end{cases} \tag{5.4}$$

Here, the counting runs only over non-negative integers, since $\sin(-m\phi)$ and $\cos(-m\phi)$ are not linearly independent from $\sin(m\phi)$ and $\cos(m\phi)$

❖ Orthogonality

The eigenfunctions solutions of this equation are orthogonal to each other when integrated over interval 2π. First, let us prove this for the complex form of the series, normalizing the functions by setting $c_m=1$:

$$\int_{-\pi}^{\pi} f_m^* f_{m'}d\phi=\int_{-\pi}^{\pi} e^{-im\phi}e^{im'\phi}d\phi=\int_{-\pi}^{\pi} e^{i(m'-m)\phi}d\phi=\begin{cases}2\pi & m=m'\\ \left.\dfrac{e^{i(m'-m)\phi}}{i(m'-m)\phi}\right|_{-\pi}^{\pi}=0 & m\neq m'\end{cases} \tag{5.5}$$

The proof for sine and cosine series is slightly more complicated. If $m\neq m'$, the sine and cosine functions can be re-expanded into terms involving $e^{\pm im\phi}$, and orthogonality follows from the above equation:

$$\begin{aligned}\int_{-\pi}^{\pi}\sin(m\phi)\sin(m'\phi)\,d\phi &= \int_{-\pi}^{\pi}\sin(m\phi)\cos(m'\phi)\,d\phi \\ &= \int_{-\pi}^{\pi}\cos(m\phi)\cos(m'\phi)\,d\phi = 0 \quad m \neq m'\end{aligned} \tag{5.6}$$

For $m = m'$, one can use the fact that $\sin(2m\phi)$ is odd on interval $[-\pi,\pi]$ to show

$$\int_{-\pi}^{\pi}\sin(m\phi)\cos(m\phi)\,d\phi = \frac{1}{2}\int_{-\pi}^{\pi}\sin(2m\phi)d\phi = 0 \tag{5.7}$$

Also, by symmetry, using the fact that sine and cosine functions are the same up to a phase shift,

$$\begin{aligned}&\int_{-\pi}^{\pi}\left(\sin^2 m\phi + \cos^2 m\phi\right)d\phi = \int_{-\pi}^{\pi}(1)d\phi = 2\pi \\ &\int_{-\pi}^{\pi}\sin^2 m\phi d\phi = \int_{-\pi}^{\pi}\cos^2 m\phi d\phi = \frac{2\pi}{2} = \pi\end{aligned} \tag{5.8}$$

Finally, normalize the $m = 0$ term to unity ($1 = \cos(0)$), giving

$$\begin{aligned}&\int_{-\pi}^{\pi}1\sin(m'\phi)\,d\phi = \int_{-\pi}^{\pi}1\cos(m'\phi)\,d\phi = 0 \\ &\int_{-\pi}^{\pi}1d\phi = 2\pi\end{aligned} \tag{5.9}$$

In quantum mechanics (QM), it is conventional to normalize the square integrated eigenfunctions to unity. However, this is not often the case in classical physics. The special functions of mathematical physics have a variety of sometimes bewildering normalizations, all of which made sense to the people who first studied them. The definition of Fourier series long predated QM. The above normalization is standard in the literature.

5.3 Definition of Fourier series

A, , function $f(x)$ which is periodic over interval $[-\pi,\pi]$, and where the positive-definite integral $\int_{-\pi}^{\pi}|f(x)|^2 dx$ is finite, can be expanded in a series of sine and cosine functions having the general form

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx) \tag{5.10}$$

The function has a finite number of maxima and minima

The function has a finite number of step-wise discrete discontinuities

The coefficients of the series are given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad \text{for } m=0,1,2\ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \quad \text{for } m=0,1,2\ldots \end{aligned} \tag{5.11}$$

The infinite series converges to the function where it is continuous and to the midpoint of the discontinuity where it is step-wise discontinuous.

❖ Completeness of the series

In what sense can the function, which may be discontinuous after all, be said to be equal to a series consisting of only continuous functions? Note the limitation of the discontinuities to a finite number of discrete points. These points have zero weight when the function is integrated. The series and the function can be said to be equivalent up to an interval of zero measure. That is,

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}\left|f(x)-f_N(x)\right|^2 dx=0 \tag{5.12}$$

where $f_N(x)$ is the partial sum of the infinite series.

❖ Sine and cosine series

If a piece-wise continuous, periodic function f(x) is an even function of x, it may be expanded in a Fourier Cosine series on interval $[-\pi,+\pi]$

$$f_c(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx) \tag{5.13}$$

If a piece-wise continuous, periodic function f(x) is an odd function of x, it may be expanded in a Fourier Sine series on interval $[-\pi,+\pi]$

$$f_s(x)=\sum_{n=1}^{\infty} b_n \sin(nx) \tag{5.14}$$

❖ Complex form of Fourier series

A real-valued function can also be represented as a complex infinite series. Let $c_{\pm m}=(a_m \mp b_m)/2$, then

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{5.15}$$

With coefficients given by

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)dx \tag{5.16}$$

Note for $m \neq 0$

$$\begin{aligned} c_m e^{imx}+c_{-m}e^{-imx} &= \left(\frac{a_m-ib_m}{2}\right)e^{imx}+\left(\frac{a_m+ib_m}{2}\right)e^{-imx} \\ &= a_m\left(\frac{e^{imx}+e^{-imx}}{2}\right)+b_m\left(\frac{e^{imx}-e^{-imx}}{2i}\right) \\ &= a_m \cos mx + b_m \sin mx \end{aligned} \tag{5.17}$$

5.4 Other intervals

Often the interval is of arbitrary length $2L$ rather than 2π . Usually it is preferable to keep the interval symmetric over $[-L, L]$. This involves making a change of variable

$$\begin{aligned} x / x' &= \pi / L \\ dx &= (\pi / L)\, dx' \end{aligned} \tag{5.18}$$

Making these changes we find

$$\begin{aligned} f(x') &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x'}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x'}{L}\right) \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x') \cos\left(\frac{n\pi x'}{L}\right) dx' \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x') \sin\left(\frac{n\pi x'}{L}\right) dx' \end{aligned} \tag{5.19}$$

5.5 Examples

❖ The Full wave Rectifier

The full wave rectifier takes a sinusoidal wave at line frequency and rectifies it using a bridge diode circuit. Figure 5-1 shows a schematic of a full wave rectifier circuit. Positive and negative parts of the line cycle take different paths through this diode bridge circuit, but the current always flows through the resister in a unidirectional manner, rectifying the signal. The result is usually filtered, by adding a capacitor to the output line, to give an approximation of a D.C. circuit, but for many purposes this first step is sufficient.

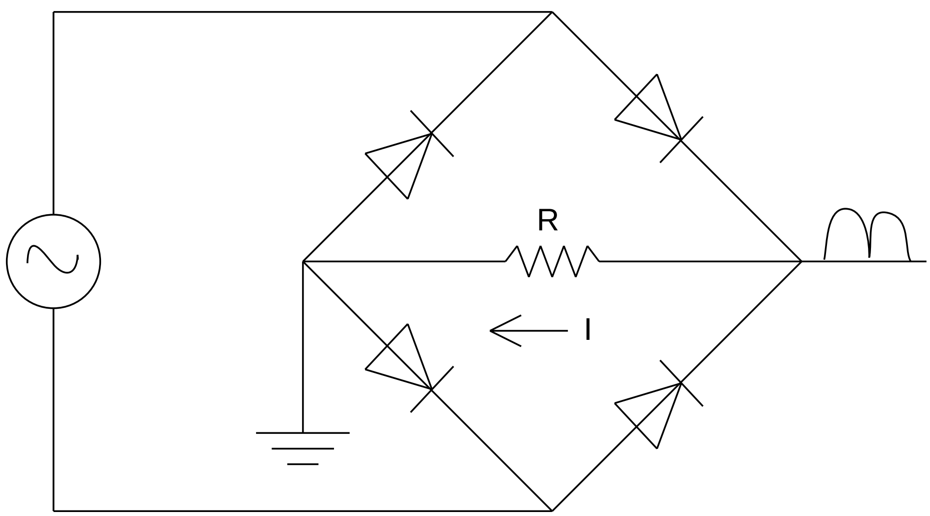


Figure 5-1 Diagram of a full wave rectifier circuit

The initial wave is a pure sine wave at the base line frequency. After rectification (Figure 5-2), this frequency disappears and one is left with a hierarchy of frequencies starting at double the base frequency. Let's look at the base wave form as it appears on an oscilloscope, locked to the line frequency. It is given by the periodic function

$$f_{line} = \sin w_o t = \sin\theta \quad -\pi < \theta \le \pi \tag{5.20}$$

where $\theta = w_o t$. After rectification, but before filtering, the modified wave form is given by

$$f_{out} = |\sin\theta| \tag{5.21}$$

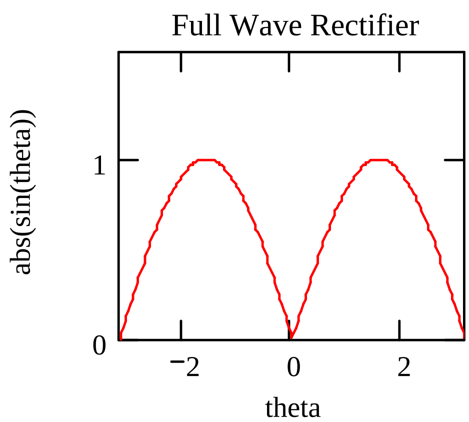


Figure 5-2 The output of a full wave rectifier, before filtering

Note that the original wave form was an odd function of θ, while the rectified function is an even function of θ. The original frequency has completely disappeared, and one is left with harmonics based on a new fundamental of double the frequency. Even symmetry under a sign change in θ implies that we can expand the solution in terms of a Fourier Cosine Series:

$$f_{out} = |\sin\theta| = \sum_{n=0}^{\infty} a_n \cos n\theta \tag{5.22}$$

The amplitudes of the frequency components are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{out} \cos n\theta d\theta \tag{5.23}$$

However, since the integrand is even, we need calculate only the positive half cycle

$$a_n = \frac{2}{\pi}\int_0^{\pi} f_{out}\cos n\theta d\theta = \frac{2}{\pi}\int_0^{\pi}\sin\theta\cos n\theta d\theta \tag{5.24}$$

This integral can easily be solved by converting to exponential notation

$$a_n = \frac{2}{\pi}\int_0^{1}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)\left(\frac{e^{in\theta}+e^{-in\theta}}{2}\right)d\theta \tag{5.25}$$

However, it is even easier to look up the answer in a book of integral tables:

$$\int_0^{\pi}\sin\theta\cos n\theta d\theta = \left(\frac{\cos\left((1-n)\theta\right)}{2(1-n)} - \frac{\cos\left((1+n)\theta\right)}{2(1+n)}\right)\Bigg|_0^{\pi} \tag{5.26}$$

The integral vanishes for odd n, and the result can be written as

$$a_n = \frac{-4}{\pi}\begin{cases}\dfrac{1}{n^2-1} & n \text{ even}\\ 0 & n \text{ odd}\end{cases} \tag{5.27}$$

The first term is positive and gives a DC offset

$$\langle f_{out}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{out}d\theta = \frac{a_0}{2} = \frac{2}{\pi} \approx 0.637 \tag{5.28}$$

$$\langle f(x)\rangle = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{a_0}{2} \tag{5.29}$$

The evaluated series can be written as

$$\left|\sin(w_0 t)\right| = \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{4\cos\left(2nw_0 t\right)}{\pi\left(\left(2n\right)^2 - 1\right)} \tag{5.30}$$

And the allowed frequencies are

$$w = 2nw_0 \tag{5.31}$$

Successively amplitudes fall off as $1/\left(\left(2n\right)^2 - 1\right)$, which means that the energy stored in the frequency components falls off as $1/\left(\left(2n\right)^2 - 1\right)^2$.

Clearly, the series is convergent since by the integral test

$$\int \left| \frac{4/\pi}{4n^2 - 1} \right| dn \xrightarrow[n\to\infty]{} \frac{1}{\pi n} \to 0 \tag{5.32}$$

By adding a capacitor on the output side of the full wave rectifier, one can short circuit the high frequency components to ground. If the capacitor is large enough, the output of the circuit is nearly D.C.

❖ The Square wave

The square wave (shown in Figure 5-4) and its variants (i.e., the step function, etc) are often found in digital circuits. The wave form is given by

$$f(\theta) = \begin{cases} +1 & 0 < \theta < \pi \\ -1 & -\pi < \theta < 0 \end{cases} \tag{5.33}$$

This function has stepwise singularities at $\theta = \{0, \pm\pi\}$. By the Fourier Series Theorem the series will converge to the midpoint of the discontinuity at those points. This function is an odd function of θ, so it can be expanded in a Fourier Sine series

$$f_{SquareWave} = \sum_{n=1}^{\infty} b_n \sin(n\theta) \tag{5.34}$$

Where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{SW} \sin n\theta d\theta = \frac{2}{\pi}\int_{0}^{\pi} f_{SW} \sin n\theta d\theta = \frac{2}{\pi}\int_{0}^{\pi} \sin n\theta d\theta \tag{5.35}$$

The solution is

$$b_n == \frac{-2}{n\pi}\int_{0}^{n\pi} d\cos n\theta = \frac{-2(\cos(n\pi) - \cos(0))}{n\pi} = \begin{cases} \dfrac{4}{n\pi} & \text{n odd} \\ 0 & \text{n even} \end{cases} \tag{5.36}$$

The frequency decomposition of the square ware is shown in Figure 5-3. The Fourier series expansion of this wave form is given by

$$f_{SW} = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin((2n+1)\theta) \tag{5.37}$$

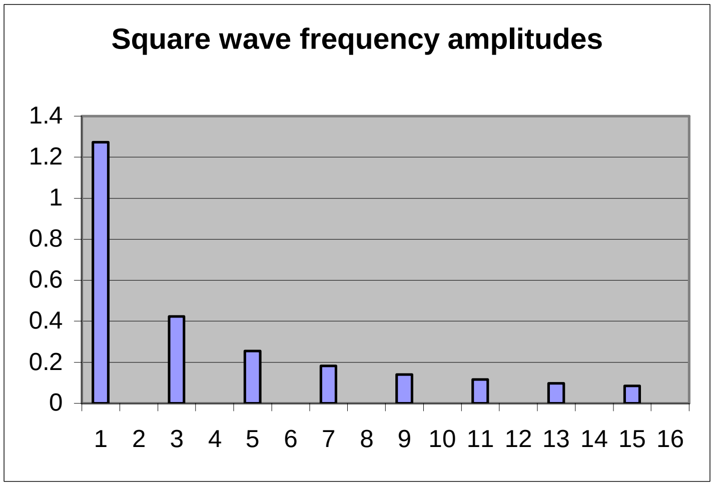


Figure 5-3 Square wave frequency components

❖ Gibbs Phenomena

The Square Wave converges slower than the first series we studied, and it is not uniformly convergent, as seen in Figure 5-4(4 and 20 terms are plotted). In fact, one can expect extreme difficulties getting a good fit at the discontinuous steps. Any finite number of terms will show in the vicinity of the discontinuity. The amplitude of this overshoots persists, but as the number of terms increases. As we approach an infinite number of terms, this overshoot covers an interval of negligible measure. This is the meaning of the expression that the series and the function are the same up to . Mathematically, this is expressed by

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}\left|f(x)-f_N(x)\right|^2 dx=0 \qquad (5.38)$$

This behavior is not unique to the square wave. Similar overshoots occur whenever there is a discontinuity. This behavior at stepwise discontinuities is referred to as the .

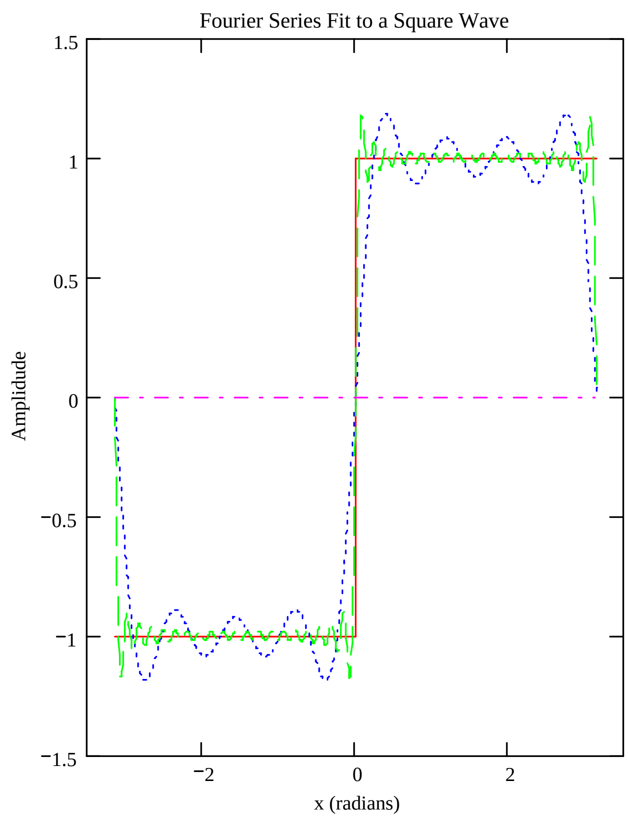


Figure 5-4 The Gibbs Phenomena

If one uses a good analog scope to view a square wave generated by a pulse generator, odds are that you won't see any such behavior. In part, this is because the analog nature of the scope. But there are more fundamental reasons. These pertain to how the signal is measured and how it was originally generated. If one has a fast pulse generator, but a slow scope, then, at the highest time resolutions, one sees a rise time in the signal due to the response of the scope. If one has a fast scope, but a limited generator, one resolves the time structure of the source instead. Piecewise step functions are not physical. They are approximations that allow us to ignore the messy details of exactly how a sudden change happened. In time dependent problems, this is called the impulse approximation.

For another example, consider a spherical capacitor, with one conducting hemisphere at positive high voltage and the second at negative high voltage. The step in voltage at the interface ignores the necessary presence of a thin insulating barrier separating the two regions. Such approximations are fine, as long as one understands their limits of validity.

Find the Fourier series expansion to the step function given by

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases} \tag{5.39}$$

Hint: Note that it can be written as a sum of an even function and an odd function.

❖ Non-symmetric intervals and period doubling

Although the interval for fitting the period is often taken to be symmetric, it need not be so. Consider the saw-tooth wave, shown in Figure 5-5, initially defined on the interval $[0, L]$.

$$f(x) = x; \quad 0 < x < L \tag{5.40}$$

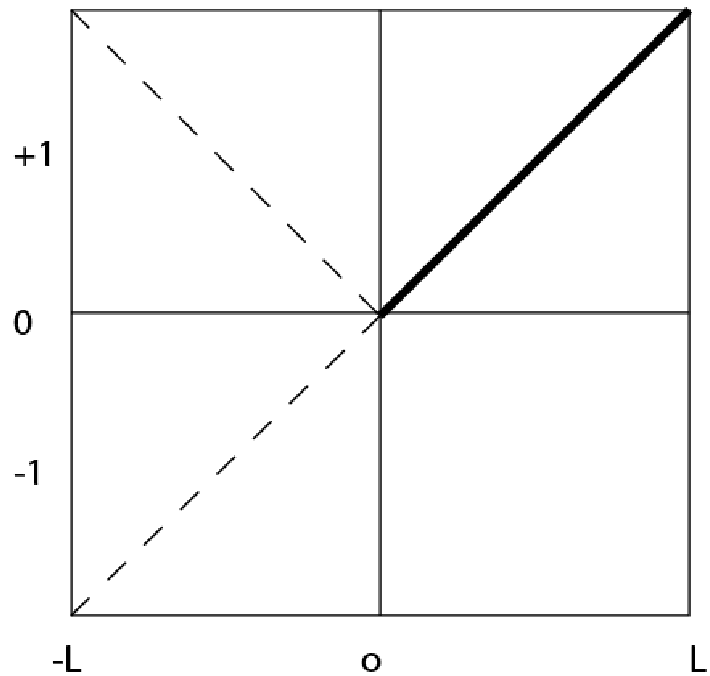


Figure 5-5 A linear wave defined on interval[0,L]

One can double the period and fit it either as n even function (Cosine Transform) or as and odd function (Sine Transform). If we are interested only a fit within this region, there are several ways of fitting this function. The most common technique is called : The interval is doubled to the interval $[-L, L]$and the function is either symmetrized or anti-symmetrized on this greater interval. The rate of convergence often depends on the choice made.

- Symmetric option (Triangle wave)

The symmetrized function represents a triangle wave of the form

$$f(x) = |x| \quad \text{for } -L < x < L \tag{5.41}$$

Fitting this with a Cosine Series, gives

$$a_n = \frac{1}{L}\int_{-L}^{L} |x| \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} x \cos\left(\frac{n\pi x}{L}\right) dx \tag{5.42}$$

Integrate by parts using $\int u dv = \int d(uv) - \int v du$ or look up in integral tables

$$\int x \cos ax dx = \frac{\cos ax}{a^2} - \frac{x \sin ax}{a} \tag{5.43}$$

This gives the solution

$$f(x) = |x| = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=0}^{\infty} \frac{\cos\left(\frac{(2n+1)\pi x}{L}\right)}{(2n+1)^2} \tag{5.44}$$

Note that after the first term, which gives the average value of the function, only terms odd in n contribute. Figure 5-6 shows that after the addition of the first cosine term, the fit to a triangle wave is already a fair approximation.

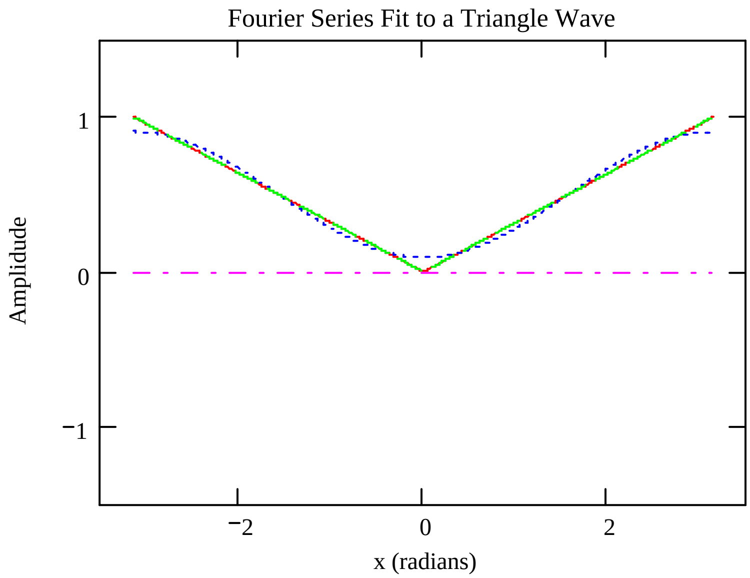


Figure 5-6 Fit to a triangle wave (2 and 10 terms)

- Antisymmetric option (sawtooth wave form)

The antisymmetrized function is a sawtooth waveform

$$f(x) = x \quad \text{for } -L < x < L \tag{5.45}$$

The solutions can now be expressed as a Fourier Sine Series

$$b_n = \frac{1}{L}\int_{-L}^{L} |x| \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx \tag{5.46}$$

Integration by parts gives

$$\int x \sin ax dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a} \tag{5.47}$$

With the solution

$$f(x) = x = \frac{2L}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right)}{n} \tag{5.48}$$

Figure 5-7 shows the Fourier series fit to the sawtooth wave form. Note that the Gibbs phenomenon has returned. Comparing the two solutions, the triangle wave converges faster $\sim \left|\frac{1}{n^2}\right|$, while the sawtooth wave converges only as $\sim \left|\frac{1}{n}\right|$. The principle difference, however, is that the triangle wave is continuous, while the saw tooth has discontinuities at $\pm\pi$. Given the choice of symmetrizing or anti-symmetrizing a wave form, pick the choice that leads to the best behaved function for the problem at hand.

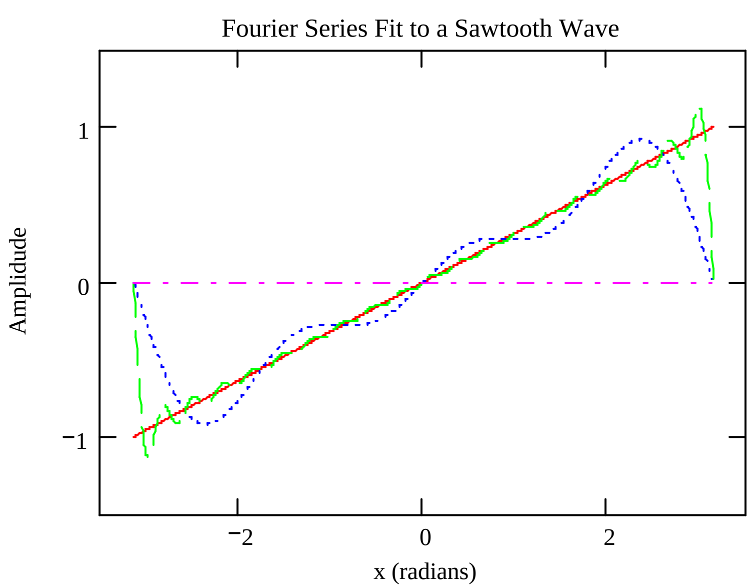


Figure 5-7 Fourier series fit to sawtooth wave(4 and 20 terms)

- Combinations of solutions

If one sums the even and odd series, the wave form remains unchanged for positive x, but cancels for negative x. This allows us solve for the function

$$f(x) = \begin{cases} x & 0 < x < L \\ 0 & -L < x < 0 \end{cases} \tag{5.49}$$

giving

$$f(x) = \frac{L}{4} - \frac{2L}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos\left(\frac{(2n+1)\pi x}{L}\right)}{(2n+1)^2} + \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \sin\left(\frac{n\pi x}{L}\right)}{n} \tag{5.50}$$

Since the function no longer has definite parity under reflection, a combined Fourier (Cosine + Sine) Series is required.

5.6 Integration and differentiation

Like Taylor series, Fourier Series can be differentiated or integrated. The effect is easiest to demonstrate using the complex form of the series.

❖ Differentiation

One can take the derivative of a series within its radius of convergence, giving

$$f'(x)=\frac{d}{dx}f(x)=\frac{d}{dx}\sum_{n=-\infty}^{\infty}c_n e^{inx}=\sum_{n=-\infty}^{\infty}(in)c_n e^{inx} \qquad (5.51)$$

Because of the added factor of n in the numerator, $f'(x)$ converges slower than $f(x)$.

❖ Integration

One can integrate a series within its radius of convergence, giving

$$\int f(x)dx=\int\sum_{n=-\infty}^{\infty}c_n e^{inx}dx=\sum_{n=-\infty}^{\infty}\frac{c_n e^{inx}}{in}+const \qquad (5.52)$$

Because of the added factor of n in the denominator $f'(x)$ converges faster than $f(x)$.

The constant of integration can be tricky. it depends on where the lower limit of integration is placed, as that affects the average value of the function. Often the integral is taken from the origin $x=0$, and the upper limit is either positive or negative x.

Example: Evaluate the integral of the square wave

$$
\begin{aligned}
f_{SW}(\theta) &= \begin{Bmatrix} +1 & x>0 \\ -1 & x<0 \end{Bmatrix} = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin\left((2n+1)\theta\right) \\
\int_0^{\theta} f(\theta) d\theta = |\theta| &= \sum_{n=1}^{\infty} \frac{4}{(2n+1)\pi} \int_0^{x} \sin\left((2n+1)\theta\right) d\theta \\
&= \sum_{n=1}^{\infty} \frac{-4}{(2n+1)^2 \pi} \cos\left((2n+1)\theta\right)\Big|_0^{\theta} \\
&= \sum_{n=1}^{\infty} \frac{-4\cos\left((2n+1)\theta\right)}{(2n+1)^2 \pi} + C
\end{aligned}
\tag{5.53}
$$

But this integral must be the triangle wave previously defined over the interval $[-\pi, \pi]$,therefore the answer should be

$$
f_{sawtooth}(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos\left((2n+1)\theta\right)}{(2n+1)^2}
\tag{5.54}
$$

Therefore, the constant of integration is

$$
C = \frac{\pi}{2}
\tag{5.55}
$$

If we didn't know the answer ahead of time, one can fix the constant by evaluation at a carefully chosen value of θ

$$
\begin{aligned}
f(0) = 0 &= \sum_{n=1}^{\infty} \frac{-4}{(2n+1)^2 \pi} + C \\
C &= \sum_{n=1}^{\infty} \frac{4}{(2n+1)^2 \pi} = \frac{\pi}{2}
\end{aligned}
\tag{5.56}
$$

Reversing the procedure, we see that the method allows us to calculate the sum of a difficult looking series of constants in closed form. This is a common use of Fourier Series.

Example: Find the value of the Zeta function $\varsigma(m)$ for m=2

The Zeta function is defined as the series of constants

$$\varsigma(m)=\sum_{n=1}^{\infty}\frac{1}{n^m} \tag{5.57}$$

Therefore

$$\varsigma(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\cdots=\frac{\pi^2}{6} \tag{5.58}$$

To prove this identity, the trick will be to find some Fourier series that gives this as a constant series for some value of its parameter. Let's try

$$f(x)=x^2 \quad -\pi<x<\pi \tag{5.59}$$

This the even series given by

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 cos(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 cos(nx)dx \\
a_0 &= \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2}{\pi}\frac{x^3}{3}\bigg|_0^{\pi} = \frac{2\pi^2}{3} \\
a_n &= \frac{2}{\pi}\left(\frac{2x}{n^2}\cos(nx)+\left(\frac{x^2}{n}-\frac{2}{n^3}\right)\sin(nx)\right)\Bigg|_o^{\pi} \\
&= \frac{4}{n^2}(-1)^n
\end{aligned} \tag{5.60}$$

or

$$f(x) = x^2 = \frac{2\pi^2}{6} + 4\sum_{n=1}^{\infty} (-1)^n \cos nx \tag{5.61}$$

Letting $x = \pi$, and using $\cos n\pi = (-1)^n$, we get

$$\begin{aligned} f(\pi) &= \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} + 4\varsigma(2) \\ \varsigma(2) &= \frac{2\pi^2}{3\cdot 4} = \frac{\pi^2}{6} \end{aligned} \tag{5.62}$$

5.7 Parseval's Theorem

We have already defined the mean (expectation) value of a Fourier series as

$$\langle f \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2} \quad -\pi < x < \pi \tag{5.63}$$

Let's now calculate the expectation value of $|f|^2$.using the complex series notation

$$\begin{aligned} \langle f^* f \rangle &= \frac{1}{2L}\int_{-L}^{L} |f|^2 \, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^* f dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n^* e^{-in\theta} \sum_{m=-\infty}^{\infty} c_m e^{im\theta} dx \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left(\int_{-\pi}^{\pi} c_n^* e^{-in\theta} c_m e^{im\theta} dx\right) \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} 2\pi\delta_{nm} c_n^* c_m = \sum_{n=-\infty}^{\infty} c_n^* c_n \ge 0 \end{aligned} \tag{5.64}$$

This is Parseval's Theorem. For classical waves, the wave's energy density is proportional to the square of the amplitude of wave. Therefore, Parseval's Theorem can be interpreted as meaning that the energy per cycle of a particular frequency is proportional to the square of its amplitude integrated over a period. In Quantum mechanics, the norm of a wave function is usually normalized to unit probability. For this case, the theorem is equivalent to saying that the partial probability of finding a particle in a frequency eigenstate n is given by the square of its amplitude.

Definition: *Parseval's Theorem*

The expectation value of the square of the absolute value of a function when averaged over its interval of periodicity is given by

$$\left\langle f^* f \right\rangle = \frac{1}{2L}\int_{-L}^{L} |f|^2 \, dx = \frac{1}{2L}\int_{-L}^{L} f^* f dx = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{5.65}$$

If the function is real-valued, then the sum can be written as

$$\left\langle f^2 \right\rangle = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \tag{5.66}$$

$|f|^2 \geq 0$ so $\left\langle f^* f \right\rangle > 0$ unless the function vanishes everywhere, except possibly on an interval of null measure. This definition is used to define the norm of a square-integrable function space. The norm of a function is just the sum of the norms of its component eigenfunctions.

❖ Generalized Parseval's Theorem

Parseval's theorem can be generalized by defining the as

$$\begin{aligned}\left\langle f^* g\right\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f^* g dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} f_n^* e^{-in\theta}\sum_{m=-\infty}^{\infty} g_m e^{im\theta} dx \\ &= \sum_{n=-\infty}^{\infty} f_n^* g_n\end{aligned} \tag{5.67}$$

5.8 Solutions to infinite series

We have already seen one way to find the sum of a series and have shown that $\varsigma(2) = \frac{\pi^2}{6}$. Here is a second way, using Parseval's Identity. Note that

$$\left\langle x^2\right\rangle = \left\langle x^* x\right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2}{\pi}\frac{x^3}{3}\bigg|_0^{\pi} = \frac{2\pi^2}{3} \tag{5.68}$$

However x is just the functional form of a sawtooth wave, and we have already solve that series. Letting the interval $L = \pi$, then the Fourier Sine series normalizes to

$$f(x) = x = \sum_{n=0}^{\infty} b_n \sin(nx), \quad b_n = \frac{(-1)^n 2}{n} \tag{5.69}$$

Then, by Parseval's Identity

$$\begin{aligned}\left\langle x^2\right\rangle &= \frac{2\pi^2}{3} = \sum_{n=1}^{\infty} b_n^2 = 4\varsigma(2) \\ \therefore \varsigma(2) &= \frac{\pi^2}{6}\end{aligned} \tag{5.70}$$

6. Orthogonal function spaces

The normal modes of the continuum
define an infinite Hilbert space.

6.1 Separation of variables

The solution of complicated mathematical problems is facilitated by breaking the problem down into simpler components. By reducing the individual pieces into a standard form having a known solution, the solution of the more complex problem can be reconstructed. For example, partial differential equations are often solvable by the technique of separation of variables. The resulting are that can be solved by the general techniques that we will explore in the following sections. The general solution to the original partial differential equation of interest can then be constructed from a summation over all product solutions to the eigenvalue equations that meet certain specified boundary requirements imposed by physical considerations.

6.2 Laplace's equation in polar coordinates

An illuminating example is the solution to in two space dimensions. The equation takes the form

$$\nabla^2\Psi = 0. \tag{6.1}$$

where ∇^2 is and Ψ can be interpreted as a static potential function in electromagnetic or gravitational theory, and as a steady state temperature in the context of thermodynamics. Let's try separating this equation in polar coordinates. The equation can be rewritten as

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right\}\Psi(r,\phi) = 0. \tag{6.2}$$

Then, we look for product solutions of the form

$$\Psi(r,\phi) = f(r)\Phi(\phi). \tag{6.3}$$

Separation of variables leads to the following coupled set ordinary differential equations

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\lambda}{r^2}\right\} f_\lambda(r) = 0. \tag{6.4}$$

$$\left\{\frac{\partial^2}{\partial \phi^2} + \lambda\right\}\Phi_\lambda(\phi) = 0. \tag{6.5}$$

We are not interested in all solutions to this eigenvalue problem, only those that make physical sense. In this case, ϕ is a cyclic variable, and the requirement that the solution be single-valued (i.e. uniquely defined) imposes the periodic boundary condition

$$\Phi(\phi + 2m\pi) = \Phi(\phi). \tag{6.6}$$

This restricts the eigenvalues to the denumerable set

$$\lambda = m^2 \geq 0; \forall \text{ integer } m = 0, \pm 1, \pm 2, \ldots. \tag{6.7}$$

The solutions, therefore, are of the form

$$\Psi(r,\phi) = \sum_{m=-\infty}^{\infty} f_m(r) e^{im\phi}. \tag{6.8}$$

For fixed r, the solution is a , which forms a complete function basis for periodic functions in ϕ.

It is easy to show (by direct substitution) that the radial solutions are

$$\begin{aligned} f_m(r) &= A_m \left(\frac{r}{r_0}\right)^m + B_m \left(\frac{r}{r_0}\right)^{-m} \quad \text{for } m \neq 0, \\ &= A_0 + B_0 \ln(r/r_0) \quad \text{for } m = 0. \end{aligned} \tag{6.9}$$

Here r_0 is some convenient scale parameter to allow the coefficients $\{A_m, B_m\}$ to all have the same units. The general solution to Laplace's equation in 2-dimensions is therefore given by

$$\Psi(r,\phi) = B_0 \ln(r/r_0) + \sum_{m=-\infty}^{\infty} \left(A_m \left(\frac{r}{r_0}\right)^m + B_m \left(\frac{r}{r_0}\right)^{-m} \right) e^{im\phi}. \tag{6.10}$$

The series solution can be interpreted as representing a multipole expansion of the potential function $\Psi(r,\phi)$.

We will return to this solution when we discuss partial differential equations in more detail in the following chapters. In particular, we will need to discuss what type of boundary conditions lead to consistent, sable, and unique solutions to the partial differential equation of interest. The product solutions, nevertheless, illustrates some general features of the separation of variable technique that are worth pointing out at this point:

- Separation of variables in partial differential equations naturally leads to ordinary differential equations that are solutions to an .
- The allowed eigenvalues are constrained by the imposed on the equations.
- The function basis generated by the eigenvalue equations forms a for the class of functions that satisfy the same boundary conditions.
- The complete solution to the partial differential equation is a to the eigenvalue equations where the coefficients are chosen to match the physical .

6.3 Helmholtz's equation

Let us next consider a class of second order partial differential equations, which contains some of the most famous named scalar equations in physics. These include

- Laplace's equation

$$\nabla^2 \Psi(\mathbf{r}) = 0. \tag{6.11}$$

- The wave equation

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial^2 t}\right)\Psi(\mathbf{r},t) = 0. \tag{6.12}$$

- The diffusion equation

$$\left(\nabla^2 - \frac{1}{\lambda}\frac{\partial}{\partial t}\right)\Psi(\mathbf{r},t) = 0. \tag{6.13}$$

- The non-interacting Schrödinger's equation

$$\left(\frac{-\hbar^2}{2m}\nabla^2 - i\hbar\frac{\partial}{\partial t}\right)\Psi(\mathbf{r},t) = 0. \tag{6.14}$$

These equations differ in their time behavior, but their spatial behavior is essentially identical. They all are linear functions of Laplace's operator ∇^2. After separating out the time behavior, they lead to a common differential equation, known as Helmholtz's equation

$$\left(\nabla^2 + k^2\right)\Psi(\mathbf{r}) = 0. \tag{6.15}$$

Or, in some cases, to the modified Helmholtz's equation

$$\left(\nabla^2 - k^2\right)\Psi(\mathbf{r}) = 0. \tag{6.16}$$

Laplace's equation refers to the special case $k^2 = 0$. The addition of a local potential term $U(\mathbf{r})$ to Helmholtz's equation changes the details of the solution, but not the general character of the boundary conditions.

The Laplacian operator is special in that it is both translationally and rotationally invariant. It is also invariant under the discrete symmetries of mirror reflection and parity reversal. As a consequence of its high degree of symmetry, separation of variables can be carried out in at least 18 commonly used coordinate frames. Here we will consider only the three most obvious ones, those employing Cartesian, spherical and cylindrical coordinates. The functions that result from its decomposition include the best known and studied equations of mathematical physics.

Helmholtz's equation can be written symbolically in the operator form

$$\left(D_x^2 + D_y^2 + D_z^2\right)\Psi = \pm k^2\Psi. \tag{6.17}$$

From which we see that the operator has an "elliptical" signature $\left(X/a\right)^2 + \left(Y/b\right)^2 + \left(Z/c\right)^2$. This signature totally determines the allowed choices of Boundary conditions.

An Elliptic Differential Equation has a unique (up to a constant), stable solution if one or the other (but not both) of the following two sets of Boundary conditions are met.

- The function is specified everywhere on a closed spatial boundary. (Dirichlet Boundary conditions), or
- The derivative of the function is specified everywhere on a closed spatial boundary (Neumann Boundary conditions).

Other choices of boundary conditions either under-specify or over-specify the constraints or lead to ambiguous or inconsistent results. The depends on the dimensionality of the equation:

- In three-dimensions, the boundary is a closed surface;
- In two dimensions, it is an enclosing line;
- In one dimension, it is given by the two end points of the line.

If the volume to be enclosed is infinite, the enclosing surface is taken as the limit as the radius $R \to \infty$ of a very large boundary envelope.

6.4 Sturm-Liouville theory

After separation of variables in the Helmholtz Equation, one is left with a set of second order ordinary differential equations having the following linear form:

$$\begin{aligned} L\{y(x)\} &= \left(A(x)\frac{d^2}{dx^2} + B(x)\frac{d}{dx} + C(x) \right) y(x) \\ &= \lambda W(x)\, y(x). \end{aligned} \tag{6.18}$$

$L = L^*$ denotes a real-valued, linear second order differential operator, A, B, C are real-valued functions of the dependent variable x, λ is an eigenvalue, $W(x)$ is a weight function, and $y_\lambda(x)$ is the eigenfunction solution to the eigenvalue equation.

This equation is assumed to be valid on a closed interval $x \subset [a,b]$. The eigenvalue solutions of the above equation are real if the linear operator can be written in the

$$L\{y(x)\}=\left(\frac{d}{dx}\left(A(x)\frac{d}{dx}\right)+C(x)\right)y(x)=\lambda W(x)y(x). \quad (6.19)$$

This requires the constraint

$$B(x)=A'(x). \quad (6.20)$$

An equation that can be put in such a form is said to be a Sturm-Liouville differential equation. If the equation is not in a self-adjoint form, an integrating factor can often be found to put it into such a form. In the standard notation for Sturm-Liouville equations the functions A and C are referred to as the functions P and Q respectively, so that the equation is often written in the standard form

$$L\{y(x)\}=\left(\frac{d}{dx}\left(P(x)\frac{d}{dx}\right)+Q(x)\right)y(x)=\lambda W(x)y(x). \quad (6.21)$$

The $W(x)$ usually arises from the Jacobean of the transformation encountered in mapping from Cartesian coordinates to some other coordinate system. This weight is required to be positive semi-definite. That is, it is except at a finite number of discrete points on the interval $x\subset[a,b]$ where it may vanish. It defines a norm for a function space such that

$$N=\int_a^b W(x)y^* y dx \geq 0 . \quad (6.22)$$

The class of functions that have a finite norm N are said to be

❖ Linear self-adjoint differential operators

A linear differential operator is said to be self-adjoint on interval $[a,b]$ if it satisfies the following criteria

$$\int_a^b y_2^* L\{y_1\}\,dx = \int_a^b y_1 L\{y_2^*\}\,dx \tag{6.23}$$

with respect to any normalizable functions y_i that meet certain specified boundary conditions at the end points of the interval.

Sturm-Liouville differential operators are self-adjoint for Dirichlet, Neumann, and periodic boundary conditions (B.C.). First note that the term $Q(x)\,\varsigma(m)$ where Q is a real-valued function is automatically self-adjoint

$$\int_a^b y_2^* Q(x) y_1 dx = \int_a^b y_1 Q(x) y_2^* dx, \tag{6.24}$$

since functions commute.

Next, integration by parts gives

$$\begin{aligned}&\int_a^b y_2^* \left(\frac{d}{dx} \left(P(x) \frac{d}{dx} y_1 \right) \right) dx \\ &= y_2^* P(x) \frac{d}{dx} y_1 \Big|_a^b - \int_a^b \frac{dy_2^*}{dx} P(x) \left(\frac{dy_1}{dx} \right) dx.\end{aligned} \tag{6.25}$$

Likewise,

$$\int_a^b y_1\left(\frac{d}{dx}\left(P(x)\frac{d}{dx}y_2^*\right)\right)dx$$
$$= y_1P(x)\frac{d}{dx}y_2^*\Big|_a^b - \int_a^b \frac{dy_2^*}{dx}P(x)\left(\frac{dy_1}{dx}\right)dx. \tag{6.26}$$

Subtracting (6.26) from (6.25) gives

$$\int_a^b y_2^* L\{y_1\}dx - \int_a^b y_1 L\{y_2^*\}dx = y_2^* P\frac{dy_1}{dx}\Big|_a^b - y_1 P\frac{dy_2^*}{dx}\Big|_a^b. \tag{6.27}$$

This clearly vanishes if the functions or their derivatives vanish at the limits $[a,b]$ (i.e., for Dirichlet or Neumann B.C.). It also vanishes if the upper and lower limits have the same value (i.e., for periodic B.C.) and also that

$$P(a) = P(b) = 0. \tag{6.28}$$

This latter case occurs for certain types of spherical functions, such as the Legendre polynomials. An important theorem is that the eigenvalues of a self-adjoint differential operator are real. The proof follows from the use of the conjugate of a Sturm-Liouville equation

$$L\{y^*(x)\} = \left(\frac{d}{dx}\left(P(x)\frac{d}{dx}\right) + Q(x)\right)y^*(x) = \lambda^* W(x)\, y^*(x). \tag{6.29}$$

(Note that the operator L and the weight function W are real). Therefore,

$$\int_a^b y_2^* L\{y_1\}dx - \int_a^b y_1 L\{y_2^*\}dx = \left(\lambda_1 - \lambda_2^*\right)\int_a^b W y_2^* y_1 dx = 0. \tag{6.30}$$

Letting $y_1 = y_2$ gives

$$\left(\lambda_1 - \lambda_1^*\right)\int_a^b W y_1^* y_1 dx = 0, \tag{6.31}$$

but the norm $\int_a^b W y_1^* y_1 dx \succ 0$ unless $y_1 \equiv 0$; therefore,

$$\lambda_1 = \lambda_1^*. \tag{6.32}$$

❖ Orthogonality

The eigenfunctions of different non-degenerate eigenvalues are orthogonal to each other with respect to weight W. The proof follows, from (6.30)

$$\left(\lambda_1 - \lambda_2\right)\int_a^b W y_2^* y_1 dx = 0. \tag{6.33}$$

If $\lambda_1 \neq \lambda_2$, this implies that

$$\int_a^b W y_2^* y_1 dx = 0. \tag{6.34}$$

Given a set of linearly independent, but degenerate, eigenfunctions ψ_n with the same eigenvalue, one can always construct a "diagonal" basis of eigenfunctions ϕ_n that are orthogonal to each other with respect to weight W. One procedure, attributed to Schmidt, is to construct the basis $\{\phi_n\}$ from the sequence

$$
\begin{aligned}
&\phi_1 = \psi_1, \\
&\phi_2 = \psi_2 - c_{21}\phi_1, \\
&\phi_3 = \psi_3 - c_{31}\phi_1 - c_{32}\phi_2, \\
&\cdots \\
&\phi_n = \psi_n - \sum_{m=1}^{n-1} c_{nm}\phi_m.
\end{aligned}
\tag{6.35}
$$

The coefficients of the n^{th} term is chosen such that ϕ_n is orthogonal to all previously orthogonalized eigenfunctions.

$$
\int_a^b W\phi_m^*\phi_n dx = 0 \quad \forall \quad m < n. \tag{6.36}
$$

For example,

$$
\begin{aligned}
\int_a^b W\phi_1^*\phi_2 dx &= \int_a^b W\phi_1^* \left(\psi_2 - c_{21}\phi_1\right) dx \\
&= \int_a^b W\phi_1^*\psi_2 dx - c_{21}\int_a^b W\phi_1^*\phi_1 dx = 0,
\end{aligned}
\tag{6.37}
$$

$$
c_{21} = \frac{\int_a^b W\phi_1^*\psi_2 dx}{\int_a^b W\phi_1^*\phi_1 dx}.
$$

When the functions are to be normalized as well as orthogonalized, this algorithm is referred to as the .

❖ Completeness of the function basis

Any piece-wise continuous function, with a finite number of maxima and minima, that is normalizable on an interval $[a,b]$ and which satisfies the same B.C. as the eigenfunctions of a self-adjoint operator on that interval can be expanded in terms of a complete basis of such eigenfunctions.

$$f(x) = \sum_{\text{all}\,\lambda} \mathrm{c}_n y_n(x) \tag{6.38}$$

If the basis is an orthogonal one with normalizations given by

$$N_n = \int_a^b W(x) y_n{}^* y_n dx \tag{6.39}$$

one can invert the problem to solve for the coefficients, giving

$$c_n = \frac{1}{N_n} \int_a^b W(x)\, f(x) y_n^* dx \tag{6.40}$$

The proof of inversion can be obtained by using the orthogonality condition

$$\int_a^b W(x) y_m{}^* y_n dx = N_n \delta_{nm} \tag{6.41}$$

❖ Comparison to Fourier Series

In retrospect, we see that our development of Fourier Series is a direct application of Sturm-Liouville Theory. Letting

$$P(x) = W(x) = 1; \quad Q(x) = 0; \quad \lambda = -m^2 \tag{6.42}$$

in the Sturm-Liouville equation and assuming periodic B.C. on interval $[-\pi, \pi]$ gives

$$\left\{\frac{\partial^2}{\partial\phi^2} + m^2\right\}\Phi_m(\phi) = 0 \tag{6.43}$$

having eigenfunctions

$$e^{im\phi}, \quad \text{for integer } m \tag{6.44}$$

in terms of which, we can expand any piece-wise continuous, normalizable, periodic function as an infinite series

$$f(x) = \sum_{m=-\infty}^{\infty} c_n e^{im\phi} \tag{6.45}$$

By orthogonality, we can solve for the coefficients giving

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi) e^{-im\phi} d\phi \tag{6.46}$$

This generalization would be a lot of work to go through just to solve a single eigenvalue equation, but it saves time in the long run. We no longer need to prove reality of the eigenvalues, orthogonality of the eigenfunctions, and completeness of the function basis in an ad-hoc manner for every eigenvalue equation that we encounter.

Discussion Problem: Generalization of Parseval's theorem

Show that Parseval's theorem can be generalized to the form

$$\int_a^b W(x)\left|f(x)\right|^2 dx = \sum_n N_n \left|c_n\right|^2. \tag{6.47}$$

And that the average of the norm of $\left|f(x)\right|^2$ with respect to weight W can be written as

$$\left\langle f^* f \right\rangle = \frac{\int_a^b W(x)\left|f(x)\right|^2 dx}{\int_a^b W(x)dx} = \sum_n \frac{N_n}{N_W}\left|c_n\right|^2, \tag{6.48}$$

where

$$N_W = \int_a^b W(x)dx. \tag{6.49}$$

If $N_n / N_W = 1$, for all eigenfunctions, the eigenfunctions are said to be normalized. The norm of a function can then be written as

$$\left\langle f^* f \right\rangle = \sum_n \left|c_n\right|^2. \tag{6.50}$$

❖ Convergence of a Sturm-Liouville series

Although we will not formally prove completeness here, it is useful to define the sense in which we mean that the function and its series expansion are equal. Partitioning the series into a finite partial sum S_N of N terms and an infinite remainder R_N, then the function and the series are the same in the sense that the norm of the remainder tends to zero as $N \to \infty$

$$\lim_{N\to\infty}\int_a^b W(x)\left|R_N(x)\right|^2 dx = \lim_{N\to\infty}\int_a^b W(x)\left|f(x)-S_n(x)\right|^2 dx = 0. \quad (6.51)$$

In plain English, this means that the series converges to the function wherever the function is continuous and the Weight function non-zero, and the function differs from the series only at a finite number of discrete points (i.e., only on intervals of null measure). This of course is essentially the same criteria that we applied to the convergence of Fourier Series.

❖ Vector space representation

Those familiar with quantum mechanics will recognize that a is just a special case of a . Hermitian operators have real eigenvalues and form complete function spaces. By treating the basis functions as representing independent degrees of freedom, one can define an infinite-dimensional vector space with coefficients $\{c_n\}$. This is, in fact, the classical origins of Hilbert space, which preceded the development of quantum mechanics. Let us expand functions with respect to a normalized eigenfunction basis:

$$\left\langle \phi_n^* \phi_m \right\rangle = \frac{\int_a^b W(x)\phi_n^*\phi_m dx}{\int_a^b W(x)dx} = \delta_{nm}. \quad (6.52)$$

Each eigenfunction can be thought of as defining an independent degree of freedom of the system, one which projects out an orthogonal state with normalization

$$\langle n|m\rangle = \delta_{nm}. \tag{6.53}$$

An arbitrary state in this infinite dimensional space can be written as

$$\begin{aligned} |f\rangle &= \sum_n c_n|n\rangle = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \end{bmatrix}, \\ \langle f| &= (|f\rangle)^\dagger = \sum_n \langle n|c_n^* = \begin{bmatrix} c_1^* & c_2^* & \dots \end{bmatrix}, \end{aligned} \tag{6.54}$$

where the basis vectors define projection operators for the amplitude coefficients

$$\langle n|f\rangle = c_n, \quad \langle f|n\rangle = c_n^*, \tag{6.55}$$

and the norm of an arbitrary vector is given by

$$\langle f|f\rangle = \sum_n |c_n|^2. \tag{6.56}$$

The connection between the and the is given by relating the inner product of the vector space with the weighted integral over the overlap of two functions.

$$\langle A|B\rangle = \sum_n a_n^* b_n = \langle A^* B\rangle = \frac{\int_a^b W(x)A^*(x)B(x)dx}{\int_a^b W(x)dx}. \tag{6.57}$$

This last expression is recognizable as the extension of Parseval's theorem applied to an arbitrary set of normalized orthogonal functions.

The interpretation of the meaning of the norm of a Hilbert space depends on the physical context. In quantum mechanics, the normalization has a probabilistic interpretation, and the total single particle wave function is normalized to unit probability. In classical wave mechanics, the square of the wave amplitude can be related to its energy density. Parseval's theorem can then be interpreted to state that normal (orthogonal) modes of oscillation contribute independently to the energy integral. The total energy of a wave is the incoherent sum of the energies of each normal mode of oscillation.

7. Spherical Harmonics

Consider a spherical cow

—Introductory physics problem example

We live on the surface of a sphere. The stars and planets are approximate spheroids, as are atomic nuclei, at the other limit of the size scale. The three-dimensional character of space, along with the assumption that one is dealing with localized sources, leads one naturally into considering using spherical coordinates. Various irreducible classes of rotational symmetries arise out of the rotational invariance of three-dimensional space. It is no surprise, then, that the spherical functions are ubiquitous in mathematical physics. Of these, the most important are the spherical harmonics $Y_{lm}(\theta,\phi)$, which represent the angular eigenfunctions of Laplace's operator in a spherical basis. These, in turn, are products of the Fourier series expansion $e^{im\phi}$ for the cyclic azimulthal angle dependence and the associated Legendre polynomial eigenfunction expansion $P_{lm}(\cos\theta)$ for polar angle behavior.

7.1 Legendre polynomials

The and their close cousins, the , arise in the solution for the polar angle dependence in problems involving spherical coordinates. The Legendre polynomials deal with the specific case where the solution is azimuthally symmetric; the associated Legendre polynomials deal with the general case. After separation of variables in the Helmholtz equation, using spherical coordinates (r,θ,ϕ) and assuming no ϕ dependence, one is left with the following differential equation

$$\frac{d}{dx}\left(1-x^2\right)\frac{d}{dx}y_l(x)=-l\left(l+1\right)y_l(x). \tag{7.1}$$

Here $x=\cos\theta$ so the domain of the equation is the interval $-1\le x\le 1$. We recognize this equation as being a Sturm-Liouville equation with $P(x)=1-x^2$, $Q(x)=0$, $W(x)=1$, having real eigenvalues $\lambda=-l(l+1)$. Because $P(\pm 1)=0$ at the end points of the interval, any piecewise-continuous, normalizable function can be expanded in a Legendre's series in the interval $[-1,1]$. By substituting $x=\cos\theta$, Legendre's equation can be written in the form

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}y_l=-l\left(l+1\right)y_l. \tag{7.2}$$

Discussion Problem: Origin of the spherical harmonics and the associated Legendre equation:

Starting with Helmholtz's equation in spherical coordinates (see Figure 7-1 for a sketch of the coordinate system)

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2}\right]\right\} = -k^2\Psi(r,\theta,\phi) \tag{7.3}$$

show that separation of variables leads to the angular equation

$$\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2}{d^2\phi}\right\}Y(\theta,\phi)=-l(l+1)Y(\theta,\phi). \tag{7.4}$$

(You don't need to solve the radial part to show this). Show by further separation of variables that

$$Y(\theta,\phi)\propto P_{lm}(x)e^{im\phi}, \tag{7.5}$$

where $P_{lm}(x)$ are the Associated Legendre polynomials given by

$$\frac{d}{dx}\left(1-x^2\right)\frac{d}{dx}P_{lm}(x)-\frac{m^2}{1-x^2}P_{lm}(x)=-l\left(l+1\right)P_{lm}(x). \tag{7.6}$$

The ordinary Legendre polynomials are related to the associated Legendre polynomials by $P_l(x)=P_{l0}(x)$

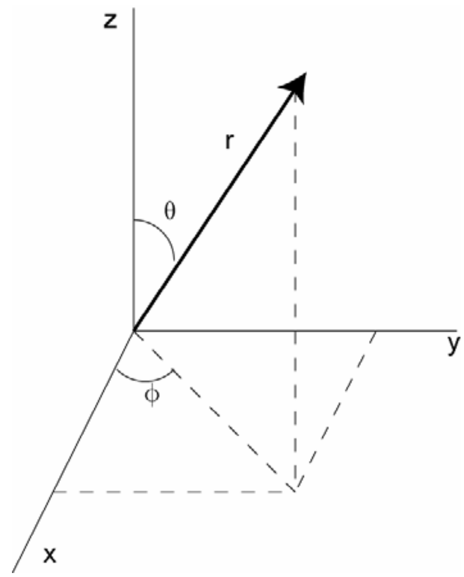


Figure 7-1 A spherical coordinate system

❖ Series expansion

Laplace's differential operator is an even function of x. Therefore, for every l, there will be two linearly-independent solutions to the eigenvalue equation that can be separated into even and odd functions. It will turn out that only one of these series will converge for the allowed values of l. Let us rewrite the equation, putting terms that couple to the same power of x on the right-hand side,

$$y'' = x^2 y'' + 2xy' - l(l+1)y \ . \tag{7.7}$$

Substituting $y = \sum_n a_n x^n$ gives the series expansion

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} l(l+1)a_n x^n \quad (7.8)$$

or

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty} [n(n+1)-l(l+1)]a_n x^n, \qquad (7.9)$$

giving the recursion relation

$$a_{n+2} = \frac{[n(n+1)-l(l+1)]}{(n+2)(n+1)}a_n, \qquad (7.10)$$

which decouples even and odd powers of x.

We can test the series to determine its radius of convergence, giving

$$|x^2| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+2}}\right| = \lim_{n\to\infty}\frac{(n+2)(n+1)}{[n(n+1)-l(l+1)]} = 1 \qquad (7.11)$$

Therefore the range is the open interval $(-1,+1)$. However, the convergence of the series at the end points is still in doubt. A more careful analysis shows that the ratio r_n approaches 1 from above for large n, and it turns out the series diverges at the end points $x = \pm 1$. This appears to be a disaster, if one fails to observe that the series terminates for integer values of l. More specifically, the even series terminates for even l, and the odd series terminates for odd l. When $n = l$, the coefficient a_{n+2} and all further terms in the series vanish, see Eq. (7.10). Therefore, the boundary conditions at $x^2 = 1$ are satisfied by setting

$$l = 0,1,2\cdots. \tag{7.12}$$

The solutions that converge at the end points of the interval are finite polynomials of order l, called the Legendre polynomials, which have an even or odd reflection symmetry given by

$$P_l(x) = (-1)^l P_l(-x) \tag{7.13}$$

For historic reasons they are normalized to 1 at $x = 1$

$$P_l(1) = 1. \tag{7.14}$$

Figure 7-2 shows a plot of the first six Legendre polynomials. By direct substitution in the recursion relation (7.10) and using the normalization constraint (7.14), the first few polynomials can be written as

$$\begin{aligned} P_0 &= 1, \\ P_1 &= x, \\ P_2 &= \tfrac{1}{2}\left(3x^2 - 1\right), \\ P_3 &= \tfrac{1}{2}\left(5x^3 - 3x\right). \end{aligned} \tag{7.15}$$

You should verify these expressions for yourselves. Let's calculate $P_2(x)$ as an example. There are two nonzero terms in the expansion, a_0 & a_2. They are related by

$$a_2 = \frac{[-l(l+1)]}{(2)(1)} a_0 = \frac{-6}{2} a_0 = -3a_0. \tag{7.16}$$

Therefore,

$$\begin{aligned} P_2(x) &= \left(-3x^2+1\right)a_0, \\ P_2(1) &= 1 \Rightarrow a_0 = -2, \\ \therefore P_2(x) &= \tfrac{1}{2}\left(3x^2-1\right). \end{aligned} \tag{7.17}$$

Note that a Legendre polynomial of order n is a power series in x of the same order n. The Legendre polynomials are bounded by

$$\left|P_l(x)\right| \le 1. \tag{7.18}$$

This can be useful in estimating errors in series expansion. A useful formula is

$$P_l(0) = \begin{cases} 0 & \text{for odd } l \\ (-1)^{l/2}\dfrac{(l-1)!!}{l!!} & \text{for even } l \end{cases} \tag{7.19}$$

❖ Orthogonality and Normalization

Since Legendre's equation is a Sturm-Liouville equation, we don't have to prove orthogonality, it follows automatically. The norm of the square-integral is given by

$$\int_{-1}^{1} P_l P_{l'} dx = \frac{2}{2l+1}\delta_{ll'}. \tag{7.20}$$

The proof will be left to a discussion problem.

A Legendre series is a series of Legendre polynomials given by

$$f\left(x\right)=\sum_{n=0}^{\infty}a_{n}P_{n}\left(x\right), \quad \left|x\right|\leq 1. \tag{7.21}$$

By orthogonality, the series can be inverted to extract the coefficients

$$a_{n}=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_{l}(x)dx. \tag{7.22}$$

A polynomial of order N can be expanded in a Legendre series of order N:

$$\sum_{m=0}^{N}b_{m}x^{m}=\sum_{n=0}^{N}a_{n}P_{n}(x). \tag{7.23}$$

The proof follows from the linear independence of the Legendre polynomials. Since a Legendre series of order N is a polynomial of order N, the above expression leads to $N+1$ linear equations relating the a_n and b_m coefficients. By linear independence, the equations have a non-trivial solution. Since a Legendre series expansion is unique, the solution obtained is the only possible solution. Solving for a_n by brute force we get

$$a_{n}=\frac{2n+1}{2}\int_{-1}^{1}P_{n}\sum_{m=0}^{N}b_{m}x^{m}dx. \tag{7.24}$$

Example: Expand the quadratic equation ax^2+bx+c in a Legendre series:

$$\begin{aligned} ax^2+bx+c &= a_0+a_1x+a_2\tfrac{1}{2}\left(3x^2-1\right) \\ &= \frac{3}{2}a_2x^2+a_1x+a_0-\frac{1}{2}a_2 . \end{aligned} \tag{7.25}$$

Therefore,

$$a_2=\frac{2}{3}a, \quad a_1=b, \text{ and } a_0=c+\frac{1}{3}a. \tag{7.26}$$

Discussion Problem: A spherical capacitor consists of two conducting hemispheres of radius r. The top hemisphere is held at positive voltage and the bottom hemisphere is held at negative voltage. The potential distribution is azimuthally symmetric and is given by

$$V(x)=\begin{cases} +V_o & \text{for } 1>x>0 \\ -V_0 & \text{for } 0>x>-1 \end{cases} \tag{7.27}$$

Calculate the Legendre series for this potential distribution.

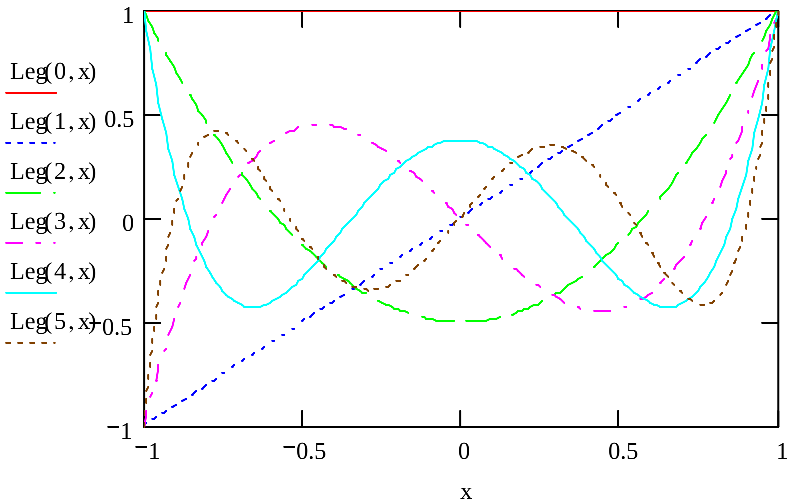


Figure 7-2 Legendre Polynomials

❖ A second solution

The second solution to Legendre's equation for integer l is an infinite series that diverges on the z-axis, where $x = \cos\theta = \pm 1$ (Figure 7-3). Although not as frequently seen, it is permitted for problems with a line-charge distribution along the z-axis. The solutions are labeled $Q_l(x)$ and have the opposite symmetry to the $P_l(x)$,

$$Q_l(x) = (-1)^{l+1} Q_l(-x) . \tag{7.28}$$

A closed form solution for $Q_l(x)$ can be found by substituting

$$Q_l(x) = \frac{1}{2} P_l(x) \ln\left(\frac{1+x}{1-x}\right) + B_l(x) \tag{7.29}$$

into Legendre's equation, where $B_l(x)$ is a second polynomial to be solved for. The first few terms are tabulated below:

$$\begin{aligned}
Q_0 &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \\
Q_0 &= \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1, \\
Q_0 &= \frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}, \\
Q_0 &= \frac{15x^3-3x}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{5x^2}{2} + \frac{2}{3}.
\end{aligned} \tag{7.30}$$

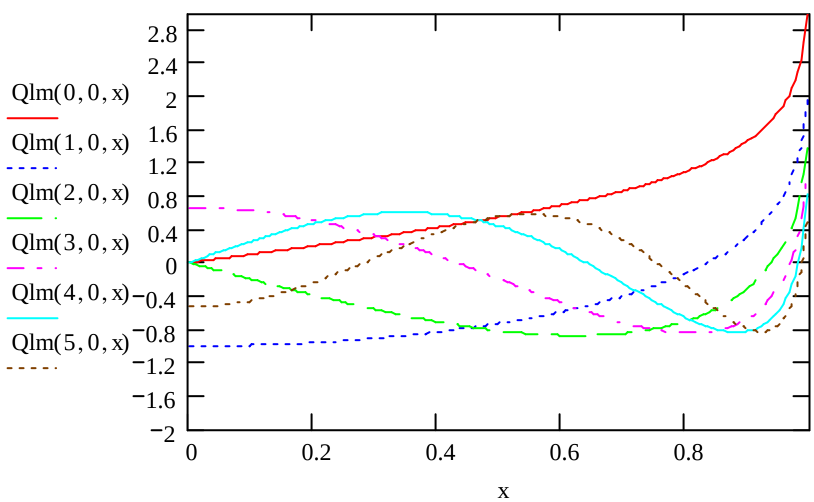


Figure 7-3 Legendre's polynomials of the second kind

Legendre polynomials are a good starting point for the study of orthogonal functions, because a number of its properties can be generalized to other orthogonal functions. There exists a differential form called Rodriquez formula that can be used to generate the polynomials. There is a generating function that serves the same purpose. Finally, there are recursion relations connecting Legendre polynomials to each other. Once one sees how these various identities apply for Legendre's polynomials, one can easily accept the existence of other such formulae for other orthogonal functions at face value, and apply them in a similar manner.

7.2 Rodriquez's formula

Rodriquez's formula for Legendre polynomials is given by

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}\left(x^2 - 1\right)^l \tag{7.31}$$

The proof uses Leibniz's rule for differentiating products.

❖ Leibniz's rule for differentiating products

The n^{th} derivative of a product of two terms is given by the binomial expansion

$$\left(\frac{d}{dx}\right)^n U(x)V(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{d^{n-m}U(x)}{dx^{n-m}} \frac{d^m V(x)}{dx^m}. \tag{7.32}$$

Proof: Let $D = d/dx$.

Then,

$$D(UV)=(DU)V+U(DV)=(D_u+D_v)UV, \tag{7.33}$$

where D_u denotes the derivative's action on the function U, and D_v denotes the derivative's action on the function V. Then

$$D^n(UV)=(D_u+D_v)^n(UV)=\left(\sum_{m=0}^{n}\binom{n}{m}D_u^{n-m}D_v^m\right)UV \tag{7.34}$$

or

$$D^n(UV)=\sum_{m=0}^{n}\binom{n}{m}\left(D_u^{n-m}U\right)\left(D_v^mV\right)=\sum_{m=0}^{n}\binom{n}{m}\left(D^{n-m}U\right)\left(D^mV\right). \tag{7.35}$$

The proof that Rodriquez's formula is correct involves

- Showing that $P_l(x)$ is a solution to Legendre's equation, and
- Showing that $P_l(1)=1$.

To prove the first part let $v=\left(x^2-1\right)^l$, then

$$\left(x^2-1\right)\frac{dv}{dx}=2lx\left(x^2-1\right)\left(x^2-1\right)^{l-1}=2lxv. \tag{7.36}$$

Differentiating this expression $l+1$ times by Leibniz's rule gives

$$\begin{gathered}D^{l+1}\left(\left(x^2-1\right)\frac{dv}{dx}\right)=D^{l+1}(2lxv)\Rightarrow\\ \binom{l+1}{0}\left(x^2-1\right)D^{l+2}v+\binom{l+1}{1}2xD^{l+1}v+\binom{l+1}{2}2D^lv\\ =l\binom{l+1}{0}2xD^{l+1}v+l\binom{l+1}{1}2D^lv,\end{gathered} \tag{7.37}$$

where

$$\binom{l+1}{0}=1,\quad \binom{l+1}{1}=l+1,\quad \binom{l+1}{2}=\frac{l(l+1)}{2} \tag{7.38}$$

which gives

$$\begin{aligned}(x^2-1)\frac{d^2}{dx^2}\left(D^l v\right)+2(l+1)x\frac{d}{dx}\left(D^l v\right)+\frac{l(l+1)}{2}2\left(D^l v\right)\\ =2lx\frac{d}{dx}\left(D^l v\right)+l(l+1)2D^l v.\end{aligned} \tag{7.39}$$

Simplifying and changing signs

$$\begin{aligned}&-(x^2-1)\frac{d^2}{dx^2}\left(D^l v\right)-2x\frac{d}{dx}\left(D^l v\right)+l(l+1)\left(D^l v\right)=0,\\ &\left(\frac{d}{dx}\left(1-x^2\right)\frac{d}{dx}+l(l+1)\right)\left(D^l v\right)=0.\end{aligned} \tag{7.40}$$

This is the Legendre equation, which completes the proof of the first part. The second part of the proof involves factoring $x^2-1=(x+1)(x-1)$ and applying Leibniz's rule to the product, then setting the result to $x=1$. Only one term in the product survives:

$$\begin{aligned}P_l(x)\Big|_{x=1}&=\frac{1}{2^l l!}(x+1)^l D^l(x-1)^l\Big|_{x=1}+\text{ terms of order }(x-1)\\ &=\frac{1}{2^l l!}(2)^l l!(x-1)^0=1.\end{aligned} \tag{7.41}$$

Example: Calculate $P_2(x)$ from Rodriquez's formula:

$$\begin{aligned} P_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}\left(x^2-1\right)^2 = \frac{1}{8}\frac{d}{dx}\left[4x\left(x^2-1\right)\right] \\ &= \frac{1}{2}\frac{d}{dx}\left(x^3-x\right) = \frac{1}{2}\left(3x^2-1\right). \end{aligned} \tag{7.42}$$

Example: Show that x^m is orthogonal to P_l if $l > m$:

Proof: One possible proof is to use Rodriquez's formula and integration by parts. Direct integration is easier. Expanding x^m in a Legendre series gives

$$\int_{-1}^{1} P_l x^m dx = \int_{-1}^{1} P_l \sum_{m'-0}^{m<l} a_{m'} P_{m'} dx = 0, \tag{7.43}$$

since $m' \neq l \; \forall m'$.

7.3 Generating function

Legendre polynomials can also be obtained by a Taylor's series expansion of the generating function

$$\Phi(x,h) = \left(1-2xh+h^2\right)^{-1/2} = \sum_{l=0}^{\infty} P_l h^l. \tag{7.44}$$

Note that, for $x = 1$, we get

$$\Phi(1,h) = \frac{1}{1-h} = \sum_{l=0}^{\infty} h^l = \sum_{l=0}^{\infty} P_l(1) h^l. \tag{7.45}$$

This will turn out to be related to why the normalization $P_l(1)=1$ was originally chosen.

Before proving (7.44) it is useful to consider its physical origin. Consider a unit charge located at a point $\mathbf{r}_0$, as shown in Figure 7-1. Its electrostatic potential is given by

$$V(\mathbf{r},\mathbf{r}_0)=\frac{q}{4\pi|\mathbf{r}-\mathbf{r}_0|}=\frac{q}{4\pi\sqrt{r^2-2xrr_0+r_0^2}}, \tag{7.46}$$

where x is the cosine of the angle between $\mathbf{r}$ and $\mathbf{r}_0$ Let

$$\begin{aligned} r_> &= Greater\left(r,r_o\right), \\ r_< &= Lesser\left(r,r_o\right), \\ h &= \frac{r_<}{r_>}\leq 1. \end{aligned} \tag{7.47}$$

Then,

$$V(\vec{r},\vec{r}_0)=\frac{q}{4\pi r_>}\Phi\left(x,h\right)=\frac{q}{4\pi r_>}\sum_{l=0}^{\infty}P_l(x)\left(\frac{r_<}{r_>}\right)^l. \tag{7.48}$$

At large distances $r\gg r_0$,The distribution approaches a pure $1/r$ potential. The generating function is the multipole expansion that arises from the fact that we didn't think to place the charge at the origin. The normalization of $P_l(1)=1$ was used to give all the angular moments equal weight at $h=1$.

Now let's turn to the proof of equation (7.44). Assume that the RHS of this equation is the definition of Φ. We want to multiply Φ by Legendre's operator

$$L = \frac{d}{dx}\left(1 - x^2\right)\frac{d}{dx}. \tag{7.49}$$

This gives

$$L\Phi = \sum_{l=0}^{\infty} L\left(P_l\right)h^l = \sum_{l=0}^{\infty} -l(l+1)P_l h^l = -h\frac{\partial^2}{\partial h^2}h\Phi \tag{7.50}$$

This results in a second order partial differential equation

$$\left(\frac{\partial}{\partial x}\left(1 - x^2\right)\frac{\partial}{\partial x} + h\frac{\partial^2}{\partial h^2}h\right)\Phi\left(x, h\right) = 0. \tag{7.51}$$

Now it is just a matter of substituting the LHS of equation (7.44) into the partial differential equation to verity that the PDE has the closed form solution: $\Phi\left(x,h\right) = \left[1 - 2xh + h^2\right]^{-1/2}$. This last step is straightforward, and is left as an exercise for the reader.

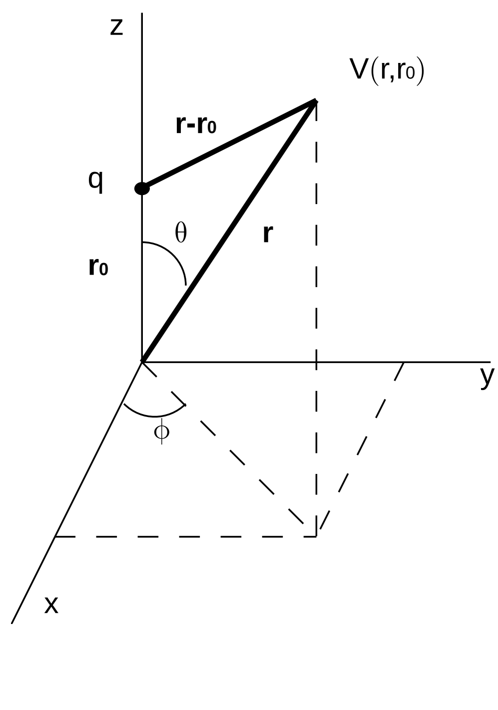


Figure 7-4 Sketch of a point charge located on the z axis

The generating function is useful for proving a number of recursion relations relating the Legendre Polynomials and their derivatives.

7.4 Recursion relations

Just like there are a number of trig identities that are useful to keep at hand, so, too, there are a number of identities relating Legendre polynomials. The first relates P_l to P_{l-1} & P_{l-2}

$$lP_l = (2l-1)xP_{l-1} - (l-1)P_{l-2} \tag{7.52}$$

Since we know that $P_0 = 1$ and $P_1 = x$, this relationship can be used recursively to generate all the other Legendre polynomials from the first two in the sequence. Unlike Rodriquez's Formula or the Generating Function, this doesn't require taking any derivatives. Other useful recursion formulas are

$$xP_l' - P_{l-1}' = lP_l, \tag{7.53}$$

$$P_l' - xP_{l-1}' = lP_{l-1}, \tag{7.54}$$

$$\left(1-x^2\right)P_l' = lP_{l-1} - lxP_l, \tag{7.55}$$

and

$$(2l+1)P_l = P_{l+1}' - P_{l-1}'. \tag{7.56}$$

Given two recursion relations, the others follow by substitution, so it is sufficient to prove the first two equations (7.52) and (7.53) are valid identities.

The first relation Eq. (7.52) can be proven by taking partial derivatives of the generating function. Taking the derivative with respect to h gives

$$
\begin{aligned}
&\frac{\partial}{\partial h}\Phi = \frac{x-h}{\left(1-2xh+h^2\right)^{3/2}} = \sum_l lP_l h^{l-1},\\
&\left(x-h\right)\Phi = \left(1-2xh+h^2\right)\sum_l lP_l h^{l-1},\\
&\left(x-h\right)\sum P_l h^l = l\left(1-2xh+h^2\right)\sum P_l h^{l-1},\\
&\sum xP_l h^l - \sum P_l h^{l+1} = \sum lP_l h^{l-1} - \sum 2lxP_l h^l + \sum lP_l h^{l+1},
\end{aligned}
\tag{7.57}
$$

Collecting terms of the same power of h, one gets the first recursion formula (7.52):

$$
\begin{aligned}
&\sum\left(2l+1\right)xP_l h^l - \sum\left(l+1\right)P_l h^{l+1} - \sum lP_l h^{l-1} = 0,\\
&\sum\left(2l-1\right)xP_{l-1}h^{l-1} - \sum\left(l-1\right)P_{l-2}h^{l-1} - \sum lP_l h^{l-1} = 0,\\
&\sum\left[\left(2l-1\right)xP_{l-1} - \left(l-1\right)P_{l-2} - lP_l h^{l-1}\right] = 0,\\
&\therefore lP_l = \left(2l-1\right)xP_{l-1} - \left(l-1\right)P_{l-2}.
\end{aligned}
\tag{7.58}
$$

In a similar manner, the solution for the second recursion relation, proceeds by taking the differential $wrt\ x$ in the definition of the generating function:

$$\frac{\partial}{\partial x}\Phi = \frac{h}{\left(1-2xh+h^2\right)^{3/2}} = \sum h^l P_l',$$

$$h\Phi = \left(1-2xh+h^2\right)\sum h^l, \tag{7.59}$$

$$\sum P_l h^{l+1} = \left(1-2xh+h^2\right)\sum h^l P'_l$$

$$= \sum P_l' h^l - 2xP_l' h^{l+1} + P_l' h^{l+2}.$$

Comparing coefficients of order h^{l+1} gives

$$P_l = P_{l+1}' - 2xP_l' + P_{l-1}'. \tag{7.60}$$

Now differentiate the first recursion relation to get

$$\begin{aligned} lP_l' &= (2l-1)P_{l-1} + x(2l-1)P_{l-1}' - (l-1)P_{l-2}' \quad \text{or} \\ (l+1)P_{l+1}' &= (2l+1)P_l + x(2l+1)P_l' - lP_{l-1}'. \end{aligned} \tag{7.61}$$

Eliminating the P_{l+1}' terms in equations (7.60) and (7.61) gives the desired result for the second recursion formula (7.53):

$$lP_l = xP_l' - P_{l-1}'. \tag{7.62}$$

Discussion Problem: Prove that $\int_{-1}^{1} P_l^2 dx = \frac{2}{2l+1}$. using the recursion relation $lP_l = xP_l' - P_{l-1}'$.

7.5 Associated Legendre Polynomials

Legendre polynomials represent the convergent solutions of the special case $m = 0$ of the associated Legendre Equation:

$$\frac{d}{dx}\left(1-x^2\right)\frac{d}{dx}P_{lm}(x)-\frac{m^2}{1-x^2}P_{lm}(x)=-l\left(l+1\right)P_{lm}(x), \qquad (7.63)$$

where

$$P_l\left(x\right)=P_{l0}\left(x\right), \qquad (7.64)$$

From the symmetry of the equation one sees that the substitution $\pm m$ leads to the same equation, therefore,

$$P_{lm}\propto P_{l-m}, \qquad (7.65)$$

Unfortunately, because of how these polynomials where originally defined, they turn out not to be simply equal to each other, they differ in their norms. They also vary in sign conventions from text to text.

Like the Legendre Polynomials, the associated Legendre functions are solutions to an eigenvalue equation of the Sturm Liouville form

$$L\left\{y\left(x\right)\right\}=\left(\frac{d}{dx}\left(P\left(x\right)\frac{d}{dx}\right)+Q\left(x\right)\right)y\left(x\right)=\lambda W\left(x\right)y\left(x\right), \qquad (7.66)$$

which differs from the Legendre equation by the addition of a function of x:

$$Q\left(x\right)=\frac{m^2}{1-x^2}, \qquad (7.67)$$

Therefore, for fixed azimulthal index m, the P_{lm} also represent eigenvalue functions of $-l(l+1)$. For positive integer $0\le m\le l$ the solutions are given by

$$P_{lm}(x) = \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_l(x), \tag{7.68}$$

which can be verified by direct substitution into the Associated Legendre equation. Substituting Rodriguez's formula for $P_l(x)$, gives the more general form,

$$P_{lm}(x) = \left(1-x^2\right)^{m/2} \frac{1}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} \left(x^2-1\right)^l. \tag{7.69}$$

This is the generalized form of Rodriquez's formula. In this form, it can be applied to both positive and negative values of m. This is in fact how the associated Legendre are defined, and gives their normalization up to a sign convention of $(-1)^m$ employed in some textbooks. Using this formula as it stands, one finds that the positive and negative m solutions are related by

$$P_{l-m}(x) = (-1)^m \left(\frac{(l-m)!}{(l+m)!}\right) P_{lm}(x), \tag{7.70}$$

and one sees that the solutions for negative m are simply proportional to those of positive m. From formula (7.68) one finds that the stretched configuration of P_{ll} is proportional to

$$P_{ll}(x) \propto \left(1-x^2\right)^{m/2} = \sin^l \theta. \tag{7.71}$$

Example: Calculate the Legendre polynomials for $l=1$

Use $P_1(x) = P_{10}(x) = x$, one needs to calculate only $P_{1\pm 1}$. Use equation (7.68) to calculate P_{11}

$$P_{11} = \left(1 - x^2\right)^{m/2} \frac{d^l}{dx^l} P_l(x) = \left(1 - x^2\right)^{1/2} \frac{d^1}{dx^1} P_1(x)$$
$$= \left(1 - x^2\right)^{1/2} \frac{d^1}{dx^1} x = \left(1 - x^2\right)^{1/2} = \sin\theta. \tag{7.72}$$

The negative values for m can be found from equation (7.70)

$$P_{1-1}(x) = (-1)^m \left(\frac{(l-m)!}{(l+m)!}\right) P_{lm}(x),$$
$$= (-1)^1 \left(\frac{(1-1)!}{(1+1)!}\right) P_{11}(x), \tag{7.73}$$
$$= \frac{-1}{2}\left(1 - x^2\right)^{1/2} = \frac{-1}{2}\sin\theta,$$

There are $(2l+1)$ $m-$states associated with a given value of l. The solution for the 3 $m-$states of P_{1m} are

$$P_{11} = \sqrt{1 - x^2} = \sin\theta,$$
$$P_{11} = x = \cos\theta, \tag{7.74}$$
$$P_{1-1} = \frac{-1}{2}\sqrt{1 - x^2} = \frac{-1}{2}\sin\theta.$$

This illustrates one of the problems with using the Associated Legendre Polynomials. Because of the—too clever by far—substitution into Rodriquez's formula, the normalization of the positive and negative m states differ. For this reason, it is better to work with the spherical harmonics directly for cases where $m \neq 0$.

❖ Normalization of Associated Legendre polynomials

The normalization of the Associated Legendre Polynomials are given by

$$\int_{-1}^{1} dx P_{lm}(x) P_{l'm}(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}. \tag{7.75}$$

Therefore a series expansion of a function of x for fixed m takes the form

$$\begin{aligned} f_m(x) &= \sum_{l=0}^{\infty} A_{lm} P_{lm}(x), \\ A_{lm} &= \frac{2l+1}{2} \frac{(l-m)!}{(l+m)!} \int_{-1}^{1} f_m(x) P_{lm}(x) dx. \end{aligned} \tag{7.76}$$

❖ Parity of the Associated Legendre polynomials

Knowing the parity of the Associated Legendre Polynomials is useful. Reflection symmetry can often be used to identify terms that identically vanish, reducing computational effort. The parity of a Legendre Polynomial of order (l, m) is

$$P_{lm}(-x) = (-1)^{l+m} P_{lm}(x). \tag{7.77}$$

Another useful result is

$$P_{lm}(\pm 1) = 0, \quad \text{for } m \neq 0. \tag{7.78}$$

❖ Recursion relations

There are a significant number of recursion relations for the Associated Legendre Polynomials. Here are a couple of examples:

- For fixed l:

$$P_{l,m+2} - \frac{2(m+1)x}{1-x^2} P_{l,m+1} + (l-m)(l+m+1)P_{l,m} = 0. \qquad (7.79)$$

- For fixed m:

$$(l+1-m)P_{l+1,m} - (2l+1)xP_{l,m} + (l+m)P_{l-1,m} = 0. \qquad (7.80)$$

This latter relation reduces to equation (7.52) when $m = 0$.

7.6 Spherical Harmonics

Legendre Polynomials do not appear in isolation. They represent the polar angle solutions to a spherical problem which also has azimulthal dependence. In particular, they are the solutions to the following angular equation which occurs when one separates Laplace's equation in spherical coordinates.

$$\left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} + l(l+1) \right\} Y_{lm}(\theta,\phi) = 0. \qquad (7.81)$$

The operator is closely related to the square of the orbital angular momentum operator in quantum mechanics:

$$L^2 Y_{lm}(\theta,\phi) = -\hbar^2 \left\{ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \right\} Y_{lm}(\theta,\phi)$$
$$= l(l+1)\hbar^2 Y_{lm}(\theta,\phi). \tag{7.82}$$

When there is no azimulthal symmetry, ($m \neq 0$), it is usually better to work directly with the product solutions $Y_{lm}(\theta,\phi)$, which are called the *spherical harmonics*. The spherical harmonics are products of the associated Legendre Polynomials and the complex Fourier series expansion of the periodic azimulthal eigenstates. They have the advantages of having a simple normalization:

$$\oint Y^*{}_{lm}(\theta,\phi) Y_{l'm'}(\theta,\phi) d\Omega = \delta_{ll'}\delta_{mm'},$$
$$\text{where } \oint d\Omega = \int_{-1}^{1} dx \int_{-\pi}^{\pi} d\phi = 4\pi. \tag{7.83}$$

The normalization of a complex Fourier series is given by

$$\int_{-\pi}^{+\pi} d\phi e^{-im\phi} e^{im'\phi} = 2\pi\delta_{mm'}. \tag{7.84}$$

while the normalization of the associated Legendre polynomials is given by equation (7.75). Putting the two together and one finds

$$Y_{lm}(\theta,\phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta) e^{im\phi}, \tag{7.85}$$

where $(-1)^m$ is a commonly used phase convention. Because different phase conventions are in common use, one has to be careful in using the spherical harmonics in a consistent manner.

This is true as well for the Associated Legendre Polynomials in general.

Any piecewise continuous function defined on a sphere $f(\theta,\phi)$ can be expressed as sums over the spherical harmonics

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{m=+l}C_{lm}Y_{lm}(\theta,\phi), \tag{7.86}$$

which, by orthogonality, gives

$$C_{lm}=\oint f(\theta,\phi)Y_{lm}^{*}(\theta,\phi)\,d\Omega. \tag{7.87}$$

If $f(\theta,\phi)$ is real, the spherical harmonics occur in complex conjugate pairs.

Calculating the spherical harmonics is not much more complicated than calculating the associated Legendre Polynomials. There are $2l+1$ m-states for every irreducible spherical harmonic tensor of rank l. For $l=0$, this reduces to a single spherically symmetric state

$$Y_{00}=\frac{1}{\sqrt{4\pi}}. \tag{7.88}$$

where it is easy to verify that

$$\oint\left|Y_{00}\right|^{2}d\Omega=1. \tag{7.89}$$

For $l=1$, equation (7.85) reduces to

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi},$$
$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta, \tag{7.90}$$
$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}.$$

Some useful formulas are

$$Y_{l,-m} = (-1)^m Y_{l,m}^* \quad \text{and} \tag{7.91}$$

$$\sum_{m=-l}^{l} \left|Y_{l,m}(\theta,\phi)\right|^2 = \frac{2l+1}{4\pi}. \tag{7.92}$$

The completeness relation for spherical harmonics is given by

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{l,m}^*(\theta',\phi') Y_{l,m}(\theta,\phi) = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi'), \tag{7.93}$$

and the multipole expansion of a point charge gives

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{l,m}^*(\theta',\phi') Y_{l,m}(\theta,\phi). \tag{7.94}$$

This latter equation is a generalization of the generating function (7.48) to the case where the point charge is not restricted to be along the z-axis.

7.7 Laplace equation in spherical coordinates

Laplace's equation in spherical coordinates can be written as

$$\nabla^2\Psi(\mathbf{r}) = \begin{pmatrix} \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \\ \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \end{pmatrix}\Psi(\mathbf{r}) \\ = 0, \tag{7.95}$$

which has product solutions of the form $\Psi(\mathbf{r}) = f_l(r)Y_{l,m}(\theta,\phi)$, yielding the radial equation

$$\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{l(l+1)}{r^2}\right)f_l(r) = 0 \quad \text{or} \\ \frac{d}{dr}r^2\frac{d}{dr}f_l(r) = l(l+1)f_l(r). \tag{7.96}$$

Letting $f_l(r) = r^\lambda$ gives $\lambda = l, -(l+1)$ leading to the solutions

$$f_l(r) = A_l\left(\frac{r}{r_0}\right)^l + B_l\left(\frac{r}{r_0}\right)^{-(l+1)}, \tag{7.97}$$

Where r_0 is a scale parameter chosen such that the coefficients *A* and *B* all have the same units. The general solution to Laplace's equation in spherical coordinates is then given by a sum over all product solutions

$$\Psi(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}\left(\frac{r}{r_0}\right)^l + B_{lm}\left(\frac{r_0}{r}\right)^{(l+1)}\right]Y_{lm}(\theta,\phi). \tag{7.98}$$

where the A_{lm} coefficients are valid for the interior solution, which includes the origin $r \to 0$, and the B_{lm} coefficients are valid for the exterior solution, which includes the point at infinity ($r \to \infty$).

8. Bessel functions

8.1 Series solution of Bessel's equation

is a solution for the radial part of the Helmholtz equation in cylindrical coordinates. The equation can be written as

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}\right)y(r)=-k^2y(r), \quad 0\le r\le\infty. \tag{8.1}$$

Letting $k^2\to -k^2$ would give us the . m is an integer for cylindrical problems, but the equation is also useful for other cases so we will replace m with the arbitrary real number p in what follows, and $y(r)$ with $J_p(x)=J_p(kr)$.

Like the sine and cosine functions in the expansion for Fourier series, the eigenvalue k can be scaled away by setting $x=kr$, and the equation can be written in the standard form

$$\left(x\frac{d}{dx}x\frac{d}{dx}-p^2+x^2\right)J_p(x)=0, \tag{8.2}$$

which can be expressed in the explicitly self-adjoint form

$$\left(\frac{d}{dx}x\frac{d}{dx}-\frac{p^2}{x}+x\right)J_p=0 \quad 0\le x\le\infty, \tag{8.3}$$

where

$$\begin{aligned} &P(x) = x, \\ &W(x) = x, \\ &Q(x) = -\frac{p^2}{x}; \\ &\lambda = 1, \end{aligned} \tag{8.4}$$

This equation can be solved by the . Noting that the operator in equation (8.2) is an even function of x. Let's try a generalized power series solution of the form

$$J_p(x) = \sum_{n=0}^{\infty} a_n \left(\frac{x}{2}\right)^{2n+s}, \tag{8.5}$$

where the factor of $2^{-(2n+s)}$ was arbitrarily inserted to simplify the normalization of the final answer. a_o is the first non-vanishing term in the series. Regroup the equation to put terms with the same power of x on the same side of the equation

$$\left(x\frac{d}{dx}x\frac{d}{dx} - p^2\right)J_p(x) = -x^2 J_p(x) \tag{8.6}$$

and substitute in the generalized power series

$$\left(x\frac{d}{dx}x\frac{d}{dx} - p^2\right)\sum_{n=0}^{\infty} a_n \left(\frac{x}{2}\right)^{2n+s} = -x^2 \sum_{n=0}^{\infty} a_n \left(\frac{x}{2}\right)^{2n+s}, \tag{8.7}$$

where expansion gives

$$\begin{aligned} &\sum_{n=0}^{\infty}\left((2n+s)^2 - p^2\right)a_n\left(\frac{x}{2}\right)^{2n+s} = -4\sum_{n=0}^{\infty} a_n \left(\frac{x}{2}\right)^{2n+s+2} \\ &= -4\sum_{n'=-1}^{\infty} a_{n-1}\left(\frac{x}{2}\right)^{2n'+s}. \end{aligned} \tag{8.8}$$

Comparing coefficients of the same power of x yields the recursion formula

$$\left((2n+s)^2 - p^2\right)a_n = \left[4n^2 + 4ns + \left(s^2 - p^2\right)\right]a_n = -4a_{n-1} \qquad (8.9)$$

Subject to the constraint that the a_{-1} term must vanish

$$\left(s^2 - p^2\right)a_0 = -4a_{-1}\,. \qquad (8.10)$$

This gives the indicial equation

$$s^2 = p^2 \qquad (8.11)$$

or

$$s = \pm p\,. \qquad (8.12)$$

In this case $\Delta s = 2p$, so if p is integer, or half-integer, there is a possibility that the two series won't be linearly independent. (This in fact is what happens for integer $p = m$, but we are getting ahead of ourselves.) Substituting into equation (8.9)

$$n(n \pm p)a_n = -a_{n-1} \qquad (8.13)$$

or

$$a_n = \frac{(-1)^n}{n!(n \pm p)!}a_0. \qquad (8.14)$$

For noninteger m, this can be written as

$$a_n = \frac{(-1)^n}{\Gamma(n+1)\Gamma(n \pm p + 1)}a_0\,. \qquad (8.15)$$

So, if we choice to normalize to $a_0 = 1$, we have the solutions

$$J_{\pm p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n \pm p+1)} \left(\frac{x}{2}\right)^{2n \pm p} \tag{8.16}$$

This is the series solution for Bessel's equation. In general, the series expansion for Bessel functions converges on the open interval $(0, \infty)$.

However, $\Gamma(p+1)$ is infinite for negative integers p, so that, for integer $p = m$, the two series are not linearly independent.

$$J_{-m}(x) = (-1)^m J_m(x). \tag{8.17}$$

❖ Neumann or Weber functions

In the case of Bessel's equation, a special technique is used to find a second linearly-independent solution. These are referred to variously in the literature as N_p or Y_p functions:

$$N_p(x) = Y_p(x) = \frac{\cos(\pi p) J_p(x) - J_{-p}(x)}{\sin(\pi p)}. \tag{8.18}$$

For noninteger p, N_p and J_p are linearly independent since $J_{\pm p}$ are linearly independent. As $p \to \mathrm{int}\, m$ one has a nonvanishing indefinite form to evaluate, which provides the second solution to Bessel's equation for integer $p = m$. Using L'Hospital's rule, the Neumann functions for integer m can be written as

$$N_m(x) = \frac{2}{\pi}\left(\ln(x/2)+\gamma\right) J_m(x) - \frac{1}{\pi}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-m} \tag{8.19}$$

where $\gamma = 0.5772156...$ is Euler's constant. This second solution is often used instead of J_{-p} even for noninteger p. The general solution to Bessel's equation is therefore given by

$$\begin{aligned} y_p(kr) &= AJ_p(kr) + BN_p(kr) \quad \text{for all } p \ge 0 \\ y_p(kr) &= AJ_p(kr) + BJ_{-p}(kr) \quad \text{for } p \ne 0,1,2,3... \end{aligned}, \tag{8.20}$$

The main difference between the Bessel and Neumann functions is that the Bessel Functions for $p \ge 0$ converge at the origin, while the Neumann functions diverge at the origin. Their respective leading order behavior for small kr is given by

$$\begin{aligned} \lim_{x\to 0} J_p(x) &= \frac{1}{\Gamma(p+1)}\left(\frac{x}{2}\right)^p + O\left(x^{p+2}\right) \\ \lim_{x\to 0} N_p(x) &= \begin{cases} \dfrac{-\Gamma(p)}{\pi}\left(\dfrac{x}{2}\right)^{-p} + O\left(x^{2-p}\right) & \text{for } p > 0 \\ \dfrac{2}{\pi}\ln(x) + O(1) & \text{for } p = 0. \end{cases} \end{aligned} \tag{8.21}$$

For large kr, the asymptotic expansions of two functions behave like phase-shifted sine and cosine functions with a decay envelop that falls of as $(kr)^{-1/2}$:

$$\lim_{x\to\infty} J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{2p+1}{4}\pi \right) + O\left(x^{-3/2}\right)$$
$$\lim_{x\to\infty} N_p(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{2p+1}{4}\pi \right) + O\left(x^{-3/2}\right). \tag{8.22}$$

8.2 Cylindrical Bessel functions

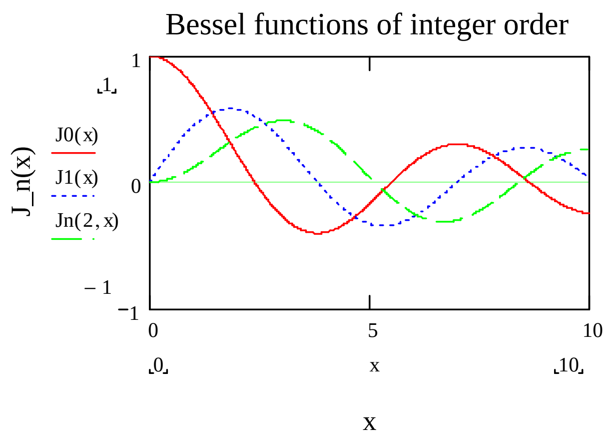


Figure 8-1 Cylindrical Bessel functions of order $m = 0, 1, 2, 3$

For integer m, the solutions to Bessel's equation are the , $J_m(kr)$ and the cylindrical Neumann functions $N_m(kr)$. Graphs for the first three Bessel functions of integer order are shown in Figure 8-1.

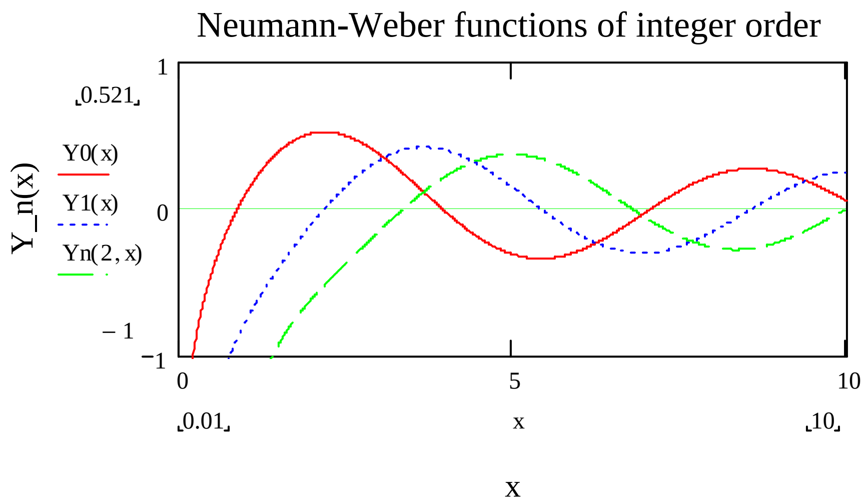


Figure 8-2 Neumann (Weber) functions of order $m = 0,1,2$

All Bessel functions for positive m, except those of order $m = 0$, start off with a zero at the origin. A similar plot showing the first few Neumann (Weber) functions is shown in Figure 8-2. Note that they diverge to negative infinity at the origin.

❖ Hankel functions

Closely related to the Bessel functions are the , which are defined by

$$\begin{aligned} H_p^{(1)}(x) &= J_p(x) + iN_p(x), \\ H_p^{(2)}(x) &= J_p(x) - iN_p(x). \end{aligned} \tag{8.23}$$

Hankel functions are most often encountered in scattering problems where the boundary conditions specify incoming or outgoing cylindrical or spherical waves.

❖ Zeroes of the Bessel functions

There are an infinite numbers of zeroes (zero-crossings) of the Bessel functions. The zeroes of the Bessel functions are important, since they provide the eigenvalues needed to find the interior solution to a cylindrical boundary value problem, where one has either Dirichlet or Neumann boundary conditions. For Dirichlet Boundary conditions, let $x = kr = ar/r_0$, where r_0 is the radius of a cylinder. Then

$$\lim_{r \to r_o} J_p\left(ar/r_0\right) = J_p\left(a\right) = J_p\left(x_{pn}\right) = 0, \tag{8.24}$$

where x_{pn} represent the n^{th} zero of the p^{th} Bessel function. Therefore the eigenvalues of $J_p\left(kr\right)$ are restricted to

$$k_{pn} = \frac{x_{pn}}{r_0}. \tag{8.25}$$

For Neumann boundary conditions one has instead

$$\lim_{r \to r_o} J'_p\left(ar/r_0\right) = J'_p\left(x'_{pn}\right) = 0, \tag{8.26}$$

where x'_{pn} represent the n^{th} zero of the derivative of the p^{th} Bessel function.

❖ Orthogonality of Bessel functions

Bessel's equation is a self-adjoint differential equation. Therefore, the solutions of the eigenvalue problem for Dirichlet or Neumann boundary conditions are orthogonal to either other with respect to the weight function $x = kr$. Like the Fourier series $f_m(\phi) = e^{im\phi}$, the eigenfunctions of Bessel's equation for fixed p are the same function $J_m(k_{nm}r)$ stretched to have a zero at the boundary.

Let a, b be distinct zeroes of J_p, then the square-integral normalization of a Bessel function is given by

$$\int_0^1 xdx\, J_p^2(ax) = \frac{1}{2} J_{p+1}^2(a). \tag{8.27}$$

Substituting $x = r/r_o$

$$\int_0^{r_0} rdr J_p^2(x_{pn}r/r_0) = \frac{r_o^2}{2} J_{p+1}^2(x_{pn}) \tag{8.28}$$

❖ Orthogonal series of Bessel functions

Consider a piecewise continuous function $f_p(r)$ that we want to expand in a Bessel function series for the interval $0 \le r \le r_o$. Let x_{pn} denote the zeroes of $J_p(x_{pn}r/r_0)$ for $r = r_o$. Then, the series expansion is given by

$$f_p(r) = \sum_{n=1}^{\infty} A_n J_p\left(x_{pn} r / r_0\right), \tag{8.29}$$

where the coefficients A_n are given by

$$\begin{aligned} A_{pn} &= \frac{2}{J_{p+1}^2\left(x_{pn}\right)} \int_0^1 x dx\, f(r) J_p\left(x_{pn} x\right) \\ &= \frac{2}{r_0^2 J_{p+1}^2\left(x_{pn}\right)} \int_0^{r_0} r dr\, f(r) J_p\left(x_{pn} r / r_0\right) \end{aligned} \tag{8.30}$$

and $x = r / r_0$.

Discussion Problem: Expand $f(r) = 1$ in a $J_0(x)$ Bessel function expansion inside a cylinder of radius a, assuming Dirichlet boundary conditions.

Note: A first glance, this problem does not appear solvable as a Bessel function series, since the function does not meet the required boundary conditions $f(r_0) = 0$. But all this really means is that we have a stepwise discontinuity at $r = a$. Orthogonal functions are well suited to handle such discontinuities. (One can expect to see some version of the Gibbs phenomena at the discontinuous point however.) Since $f(r)$ is nonzero at $r = 0$, it is appropriate to try an expansion in terms of $J_0(x_{0n} r / a)$. (Expansions in $J_p(x)$ for $p \neq 0$ would not work.) The function to be fitted can by rewritten as

$$\begin{aligned} f(r) &= 1 \text{ for } r < a, \\ f(r) &= 0 \text{ for } r = a. \end{aligned} \tag{8.31}$$

Example: Plot the first few eigenfunctions of $J_0(kr)$ that have zeroes at $r/r_0 = 1$.

The zeroes of the Bessel functions are transcendental numbers. One can find their values numerically, using a root finding algorithm. This gives the values following values for the roots of $J_0(x)$:

$$x_{0n} = \{2.405,\ 5.52,\ 8.654,\ 11.792,\ 14.931,\ 18.071, \cdots\}. \qquad (8.32)$$

Figure 8-3 shows the first four eigenfunctions of $J_0(kr)$ satisfying Dirichlet Boundary conditions at $r = 1$, $(k_{mn} = x_{mn})$

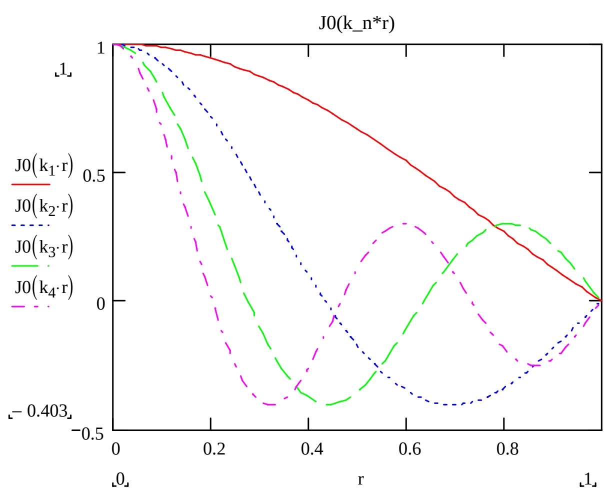


Figure 8-3 First four eigenfunctions of $J_0(kr)$ satisfying Dirichlet boundary conditions at r=1.

For fixed p, a function $f_p(r)$ that is finite at the origin and vanishes at cylindrical boundary $r = r_0$ () can be expanded as a Bessel function series

$$f_p(r) = \sum_{n=1}^{\infty} A_{pn} J_p\left(x_{pn} r / r_0\right), \\ f_p(r_0) = 0. \tag{8.33}$$

If, instead, one were to use , one would use the expansion

$$f_p(r) = \sum_{n=1}^{\infty} A_{pn} J_p\left(x'_{pn} r / r_0\right), \\ f'_p(r_0) = 0. \tag{8.34}$$

❖ Generating function

The generating function for Bessel functions of integer order is given by

$$e^{m(t-1/t)/2} = \sum_{=-\infty}^{\infty} J_m(x) t^m . \tag{8.35}$$

❖ Recursion relations

Like Legendre Polynomials, there are a large number of useful recursion formulas relating Bessel functions. Some of the more useful identities are

$$J_{p+1}(x) = \frac{2}{p} J_p(x) - J_{p-1}(x) , \tag{8.36}$$

$$J'_p(x) = \frac{1}{2}\left[J_{p-1}(x) - J_{p+1}(x)\right], \tag{8.37}$$

and

$$J'_p(x) = -\frac{p}{x}J_p(x) + J_{p-1}(x) = \frac{p}{x}J_p(x) - J_{p+1}(x). \tag{8.38}$$

Of particular importance are the raising and lowering ladder operators that relate a Bessel function to the next function on the ladder:

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x) \tag{8.39}$$

and

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = x^{-p} J_{p+1}(x). \tag{8.40}$$

The Neumann functions satisfy the same relations as (8.36)-(8.40).

These recursion relations can most readily be proven by direct substitution of the series expansion given by equation (8.16). For example, the proof of equation (8.39) is given by

$$\begin{aligned}\frac{d}{dx}\left[x^p J_p(x)\right] &= \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)}\left(\frac{x^{2n+2p}}{2^{2n+p}}\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+2p)}{\Gamma(n+1)\Gamma(n+p+1)}\left(\frac{x^{2n+2p-1}}{2^{2n+p}}\right).\end{aligned} \tag{8.41}$$

Using $\Gamma(n+p+1) = (n+p)\Gamma(n+p)$ gives

$$\frac{d}{dx}\left[x^p J_p(x)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p)} \left(\frac{x^{2n+2p-1}}{2^{2n+p-1}}\right)$$
$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p)} \left(\frac{x^{2n+p-1}}{2^{2n+p-1}}\right) \tag{8.42}$$
$$= x^p J_{p-1}.$$

8.3 Modified Bessel functions

If one makes the replacement of $k \to ik$ in Bessel's equation (8.1), one gets the modified Bessel equation.

$$\left(x\frac{d}{dx}x\frac{d}{dx} - p^2 - x^2\right) I_p(x) = 0. \tag{8.43}$$

where $I_p(x)$ denote the modified Bessel functions of the first kind. Their series solution is nearly identical to Bessel's series (8.16), except that the coefficients no longer alternate in sign

$$I_{\pm p}(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)\Gamma(n \pm p+1)} \left(\frac{x}{2}\right)^{2n \pm p}. \tag{8.44}$$

Noting that the substitution $k \to ik$ is equivalent to the substitution $x \to ix$, so the solutions also can be written as

$$I_p(x) = i^p J_p(ix), \tag{8.45}$$

where the factor i^p is included so that the series expansion (8.44) is a real-valued function.

If the Bessel functions could be said to be oscillatory in character, asymptotically involving decaying sinusoidal functions, the solutions to the modified Bessel equation are exponential in behavior. For positive p, the solutions are finite at the origin and grow exponentially with increasing x as shown in Figure 8-4. They have the asymptotic behavior

$$I_p(x) \sim \sqrt{\frac{2}{\pi x}} e^x \text{ for large } x. \tag{8.46}$$

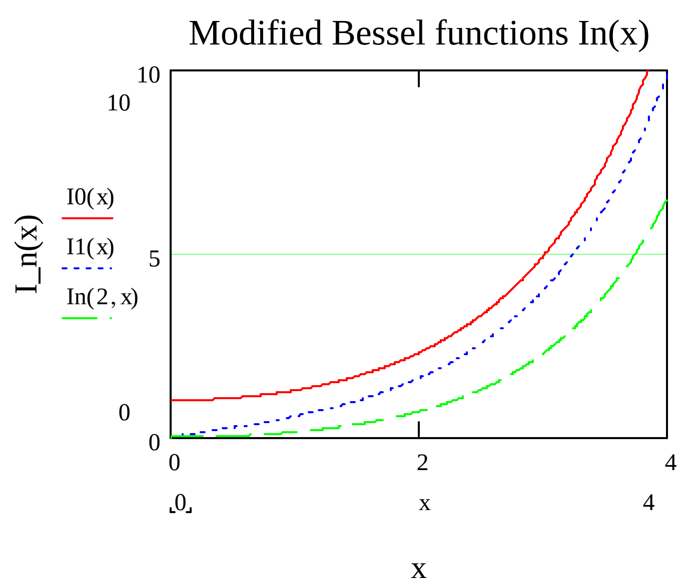


Figure 8-4 Modified Bessel Functions of the first kind

For integer $p \to m$, the solutions $I_{\pm p}(x)$ are not linearly independent,

$$I_{-m}(x) = I_m(x). \tag{8.47}$$

❖ Modified Bessel functions of the second kind

To obtain a second linearly-independent solution, valid for all p, the linear combination

$$K_p(x) = \frac{\pi}{2\sin \pi p}\left[I_{-p}(x) - I_p(x)\right] \tag{8.48}$$

is used. The modified Bessel functions of the second kind $K_p(x)$ diverge at the origin. They exponentially decay for large values of x, as shown in Figure 8-5, with the asymptotic behavior

$$K_p(x) \sim \sqrt{\frac{1}{2\pi x}}e^{-x} \text{ for large } x. \tag{8.49}$$

For integer $p = m$, the series expansion for $K_p(x)$, calculated using L'Hospital's rule, is given by

$$\begin{aligned} K_m(x) = (-1)^m\left[\ln(x/2) + \gamma\right]I_m(x) + \frac{1}{2}\sum_{k=0}^{m-1}(-1)^k(m-k-1)!\left(\frac{x}{2}\right)^{2k-m} \\ + \frac{(-1)^m}{2}\sum_{k=0}^{m-1}\frac{\Phi(k)+\Phi(k+m)}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m}, \end{aligned} \tag{8.50}$$

where

$$\begin{aligned} &\Phi(n) = \sum_{n'=1}^{n}\frac{1}{n'} \quad \text{for } n' \neq 0, \\ &\Phi(0) = 0. \end{aligned} \tag{8.51}$$

For small x, $I_p(x)$ and $K_p(x)$ have the leading order expansions

$$
\begin{aligned}
I_p(x) &= \frac{1}{\Gamma(p+1)}\left(\frac{x}{2}\right)^p + O\left(x^{-p+2}\right), \\
K_p(x) &= \begin{cases} \frac{1}{2}\Gamma(p)\left(\frac{x}{2}\right)^{-p} + \begin{cases} O\left(x^{2-p}\right) & \text{for } p>1, \\ O\left(x^p\right) & \text{for } 0<p<1, \end{cases} \\ -\ln(x) + O(1) & \text{for } p=0. \end{cases}
\end{aligned} \tag{8.52}
$$

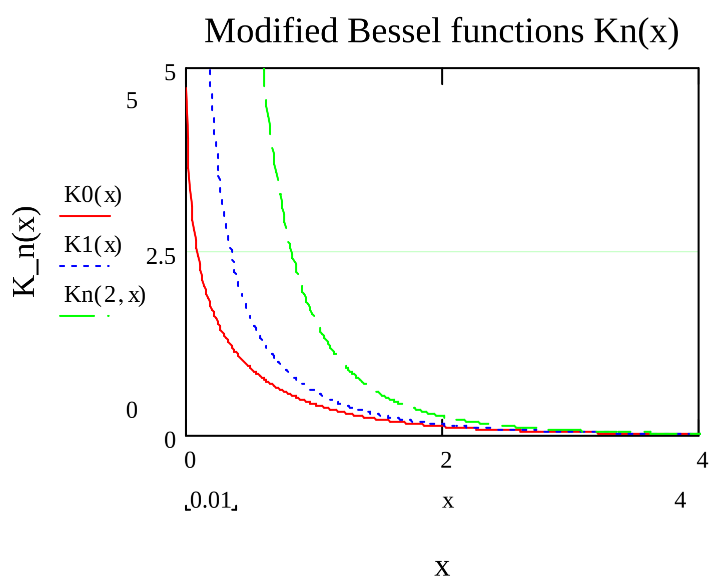


Figure 8-5 Modified Bessel functions of the second kind

❖ Recursion formulas for modified Bessel functions

Unlike their close cousins, the Bessel functions of the first and second kind, the modified Bessel functions of the first and second kind satisfy different recursion formulas. Several of the more useful of these are listed below, others can be found in standard compilations of mathematics tables.

$$\begin{aligned} I_{p+1}(x) &= I_{p-1}(x) - \frac{2}{p} I_p(x), \\ K_{p+1}(x) &= K_{p-1}(x) + \frac{2}{p} K_p(x), \end{aligned} \tag{8.53}$$

$$\begin{aligned} I'_p(x) &= \frac{1}{2}\left[I_{p-1}(x) + I_{p+1}(x) \right], \\ K'_p(x) &= -\frac{1}{2}\left[K_{p-1}(x) + K_{p+1}(x) \right], \end{aligned} \tag{8.54}$$

$$\begin{aligned} \frac{d}{dx}\left[x^p I_p(x) \right] &= x^p I_{p-1}(x), \\ \frac{d}{dx}\left[x^p K_p(x) \right] &= -x^p K_{p-1}(x), \end{aligned} \tag{8.55}$$

$$\begin{aligned} \frac{d}{dx}\left[x^{-p} I_p(x) \right] &= x^{-p} I_{p+1}(x), \\ \frac{d}{dx}\left[x^{-p} K_p(x) \right] &= -x^{-p} K_{p+1}(x). \end{aligned} \tag{8.56}$$

8.4 Solutions to other differential equations

A significant use of Bessel's functions is in finding the solutions of other differential equations. For example, the second order differential equation of the form

$$y''(x) + \frac{1-2a}{x} y'(x) + \left[\left(bcx^{c-1} \right)^2 + \frac{a^2 - p^2c^2}{x^2} \right] y(x) = 0 \tag{8.57}$$

has the solution

$$y = x^a Z_p\left(bx^c\right), \tag{8.58}$$

where Z_p is any linear combination of the Bessel functions J_p and N_p, and a, b, c, p are constants.

8.5 Spherical Bessel functions

The spherical Bessel equation represents the radial solution to the Helmholtz equation in spherical coordinates. This equation can be written as

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right]y(r) = 0 \tag{8.59}$$

where the values of $l = 0, 1, 2\cdots$ is restricted to integer values. The substitution $x = kr$ is made to put the equation into dimensionless form and to scale away the eigenvalue k^2. The resulting equation can be rewritten in the self-adjoint form as

$$\left[\frac{d}{dx}x^2\frac{d}{dx} - l(l+1) + x^2\right]y(r) = 0. \tag{8.60}$$

Note that the equation has a weight factor $W(x) = x^2 = (kr)^2$. This factor of r^2 in the weight comes from the Jacobean of transformation of an element of volume when expressed in spherical coordinates

$$dV = r^2 dr d\Omega. \tag{8.61}$$

The solution to the equation can be given in terms of elementary functions, but it is usual to express the solution in terms of Bessel functions. Using (8.57) and (8.58) one finds the solution

$$y_l(x) = x^{-1/2} Z_{(l+1/2)}(x) \tag{8.62}$$

(try $a = -1/2, b = c = 1, p = l + 1/2$).

Discussion Problem: Show by mathematical induction that

$$j_l(x) = x^l \left(\frac{-d}{xdx}\right)^l j_o(x), \tag{8.63}$$

where

$$j_o(x) = \sin x / x, \tag{8.64}$$

is a solution to the spherical Bessel equation (8.60).

❖ Definitions

The spherical Bessel functions of the first and second kind are defined as

$$\begin{aligned} j_l(x) &= \sqrt{\frac{\pi}{2x}} J_{(l+1/2)}(x) \\ &= x^l \left(\frac{-d}{xdx}\right)^l \frac{\sin x}{x}, \end{aligned} \tag{8.65}$$

$$n_l(x) = \sqrt{\frac{\pi}{2x}} N_{(l+1/2)}(x) \\ = x^l \left(\frac{-d}{xdx} \right)^l \frac{-\cos x}{x}. \tag{8.66}$$

Like the cylindrical Bessel functions, the spherical Bessel functions of the first (Figure 8-6) and second (Figure 8-7) kind are oscillatory, with an infinite number of zero crossings. For large x their decay envelope falls off as $1/x$.

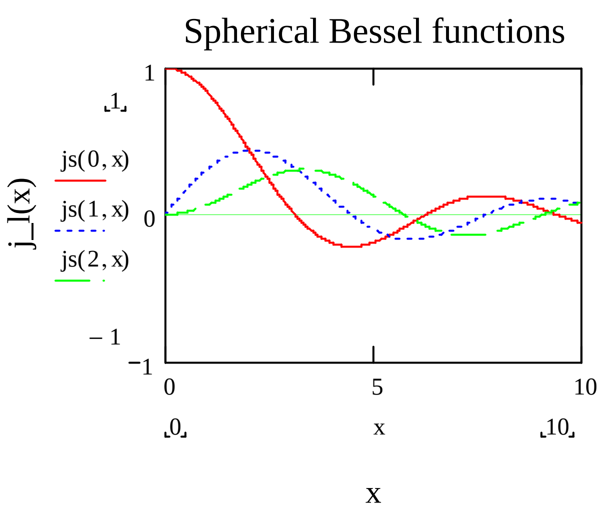


Figure 8-6 Spherical Bessel functions

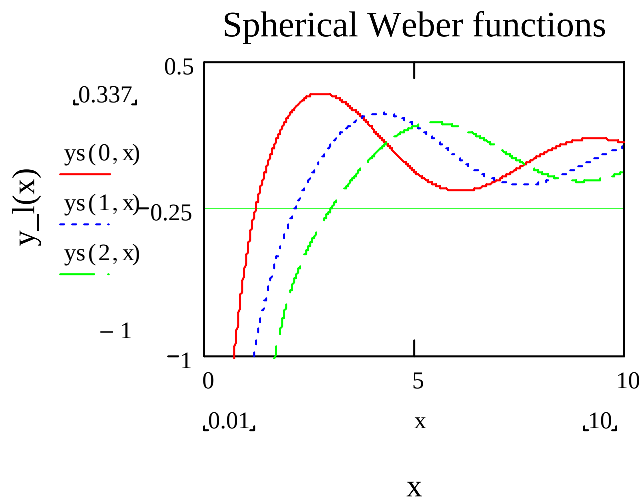


Figure 8-7 Spherical Neumann (Weber) functions

The spherical Hankel functions are defined as

$$h_l^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H_{l+1/2}^{(1)}(x) = j_l(x) + i n_l(x), \tag{8.67}$$

$$h_l^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H_{l+1/2}^{(2)}(x) = j_l(x) - i n_l(x). \tag{8.68}$$

Lastly, the modified spherical Bessel functions are given by

$$\begin{aligned} i_l(x) &= \sqrt{\frac{\pi}{2x}} I_{(l+1/2)}(x) \\ &= x^l \left(\frac{d}{x dx} \right)^l \frac{\sinh x}{x}, \end{aligned} \tag{8.69}$$

$$\begin{aligned} k_l(x) &= \sqrt{\frac{\pi}{2x}} K_{(l+1/2)}(x) \\ &= x^l \left(\frac{-d}{x dx} \right)^l \frac{e^{-x}}{x}. \end{aligned} \tag{8.70}$$

Table 8-1 lists the first three l values of the most common spherical Bessel functions. The limiting behavior of these functions for small and large values of x are summarized in Table 8-2.

Table 8-1 Spherical Bessel functions of order 0, 1, and 2

l	0	1	2
$j_l(x)$	$\frac{\sin x}{x}$	$\frac{\sin x - x\cos x}{x^2}$	$\frac{\left(3-x^2\right)\sin x - 3x\cos x}{x^3}$
$n_l(x)$	$-\frac{\cos x}{x}$	$\frac{-\cos x - x\sin x}{x^2}$	$\frac{-\left(3-x^2\right)\cos x - 3x\sin x}{x^3}$
$i_l(x)$	$\frac{\sinh x}{x}$	$\frac{x\cosh x - \sinh x}{x^2}$	$\frac{\left(x^2+3\right)\sinh x - 3x\cosh x}{x^3}$
$k_l(x)$	$\frac{e^{-x}}{x}$	$\frac{e^{-x}}{x}\left(x+1\right)$	$\frac{e^{-x}}{x}\left(x^2+3x+3\right)$

Table 8-2 Asymptotic limits for spherical Bessel Functions

l	$x \ll 1$	$x \gg 1$
$j_l(x)$	$\approx \dfrac{x^l}{(2l+1)!!}$	$\sim \dfrac{1}{x}\sin(x - n\pi/2)$
$n_l(x)$	$\approx \dfrac{-(2l-1)!!}{x^{l+1}}$	$\sim \dfrac{-1}{x}\cos(x - n\pi/2)$
$i_l(x)$	$\approx \dfrac{x^l}{(2l+1)!!}$	$\sim \dfrac{e^x}{2x}$
$k_l(x)$	$\approx \dfrac{(2l+1)!!}{x^{l+1}}$	$\sim \dfrac{e^{-x}}{x}$

❖ Recursion relations

Some recursion relations for the spherical Bessel functions are summarized in (8.71) where f_l can be replaced by any of the functions $j_l, n_l, h_l^{(1)}, h_l^{(2)}$.

$$
\begin{aligned}
& f_{l-1}(x) + f_{l+1}(x) = \frac{2l+1}{x} f_l(x), \\
& n f_{l-1}(x) - (l+1) f_{l+1}(x) = (2l+1)\frac{d}{dx} f_l(x), \\
& \frac{d}{dx} f_l(x) = f_{l-1}(x) - \frac{l+1}{x} f_l(x) = -f_{l+1}(x) + \frac{l}{x} f_l(x).
\end{aligned}
\tag{8.71}
$$

The ladder operators for the spherical Bessel functions are given by

$$\frac{d}{dx}\left[x^{l+1} f_l(x)\right] = x^{l+1} f_{l-1}(x),$$
$$\frac{d}{dx}\left[x^{-l} f_l(x)\right] = -x^{-l} f_{l+1}(x). \tag{8.72}$$

The equivalent recursion relations for the modified spherical Bessel functions are summarized in (8.73) where f_l can be replaced by i_l or $(-1)^{l+1}k_l$.

$$f_{l-1}(x) - f_{l+1}(x) = \frac{2l+1}{x} f_l(x),$$
$$nf_{l-1}(x) + (l+1) f_{l+1}(x) = (2l+1)\frac{d}{dx} f_l(x), \tag{8.73}$$
$$\frac{d}{dx} f_l(x) = f_{l-1}(x) - \frac{l+1}{x} f_l(x) = f_{l+1}(x) + \frac{l}{x} f_l(x).$$

The ladder operators for the modified spherical Bessel functions are given by

$$\frac{d}{dx}\left[x^{l+1} f_l(x)\right] = x^{l+1} f_{l-1}(x),$$
$$\frac{d}{dx}\left[x^{-l} f_l(x)\right] = x^{-l} f_{l+1}(x). \tag{8.74}$$

❖ Orthogonal series of spherical Bessel functions

Let $x = r/r_0$, where r_0 is the surface of a sphere. Assuming Dirichlet boundary conditions, the eigenfunctions of the spherical Bessel functions that vanish on this surface are given by

$$j_l\left(a_{l,n}\right) = 0\,, \tag{8.75}$$

where $a_{l,n}$ denotes the n^{th} zero of the l^{th} spherical Bessel function. Suppose one has a function $f_l(r)$ defined on the interior of this sphere. Assume one wants to expand $f_l(r)$ in a Bessel function series of order l. The expansion would take the form

$$f_l(r) = \sum_{n=0}^{\infty} A_{l,n} j_l\left(a_{l,n} r / r_0\right). \tag{8.76}$$

(Figure 8-8 shows how the functions scale to fit in the nth zero at the boundary) Since this is a series of orthogonal functions, one can use the orthogonality relation, which is given by

$$\int_0^1 x^2 dx j_l\left(ax\right) j_l\left(bx\right) = \frac{j_{l+1}^2(a)}{2} \delta_{ab}, \tag{8.77}$$

where a, b denote two zeroes of the l^{th} Bessel function. Therefore, the coefficients $A_{l,n}$ are given by

$$\begin{aligned} A_{l,n} &= \frac{2}{j_{l+1}^2(a)} \int_0^1 x^2 dx f_l(r) j_l(a_{l,n} r / r_0) \\ &= \frac{2}{j_{l+1}^2(a) r_0^3} \int_0^1 r^2 dr f_l(r) j_l(a_{l,n} r / r_0). \end{aligned} \tag{8.78}$$

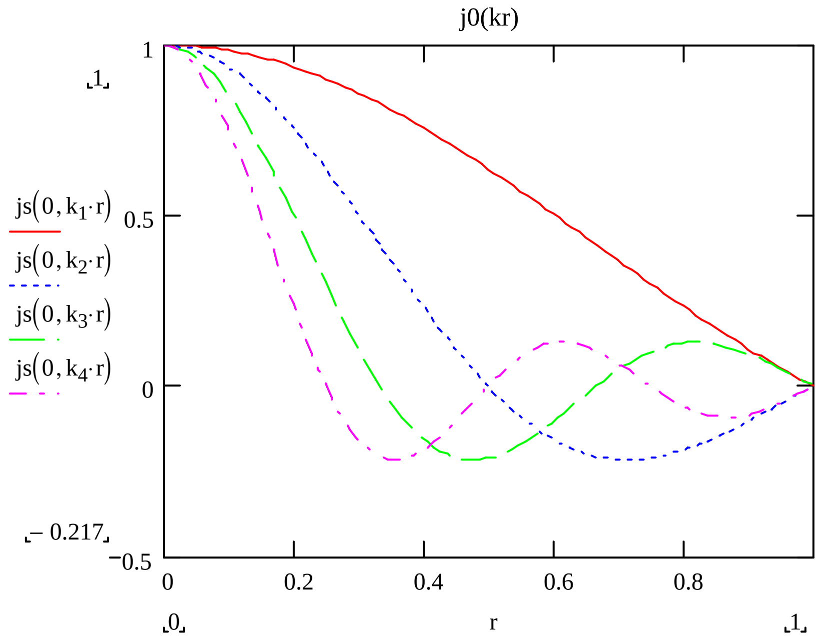


Figure 8-8 Eigenfunctions of $j_0(kr)$

Example: Expand, in a series of spherical Bessel functions, the distribution

$$f(r) = \begin{cases} f_0 & \text{for } 0 \le r < r_0 \\ 0 & \text{for } r = r_0 \end{cases} \tag{8.79}$$

Where the series solution is valid for $r \le r_0$.

The distribution is spherically symmetric, so one can expand the function in a series of $l = 0$ Bessel functions

$$f(r) = \sum_{n=1}^{\infty} A_n j_0\left(a_{0,n} r / r_0\right), \tag{8.80}$$

where

$$A_n = \frac{2f_0}{j_1^2(a_n)} \int_0^1 x^2 dx j_0(a_{0,n}x) \,. \tag{8.81}$$

Using the ladder operators (8.72)

$$\frac{d}{dx}\left[x^2 j_1(x)\right] = x^2 j_0(x) \,, \tag{8.82}$$

the integral can be evaluated, giving

$$\begin{aligned} \int_0^1 x^2 dx j_0(a_n x) &= \frac{1}{a_{0,n}^3} \int_0^{a_{0,n}} x'^2 dx' j_0(x') \\ &= \frac{1}{a_{0,n}^3} \left[x^2 j_1(x)\right]_0^{a_{0,n}} = \frac{1}{a_{0,n}} j_1\left(a_{0,n}\right), \end{aligned} \tag{8.83}$$

$$A_n = \frac{2f_0}{j_1^2(a_n)} \frac{j_1\left(a_{0,n}\right)}{a_{0,n}} = \frac{2f_0}{a_{0,n} j_1(a_n)} \,, \tag{8.84}$$

yielding the result

$$f\left(r\right) = 2f_0 \sum_{n=1}^{\infty} \frac{j_0\left(a_{0,n} r / r_0\right)}{a_{0,n} j_1(a_{0,n})} \,. \tag{8.85}$$

The zeros of j_0 are given by $a_{0,n}=n\pi$. The results of the series approximation are shown in Figure 8-9. Because the distribution is discontinuous, the overshooting effect that is characteristic of the Gibbs Phenomena is observed. The magnitude of the overshoot persists even when increasing the number of terms, but the area of the overshoot gets smaller. In the infinite series limit, the series and the function would agree except on a interval of null measure.

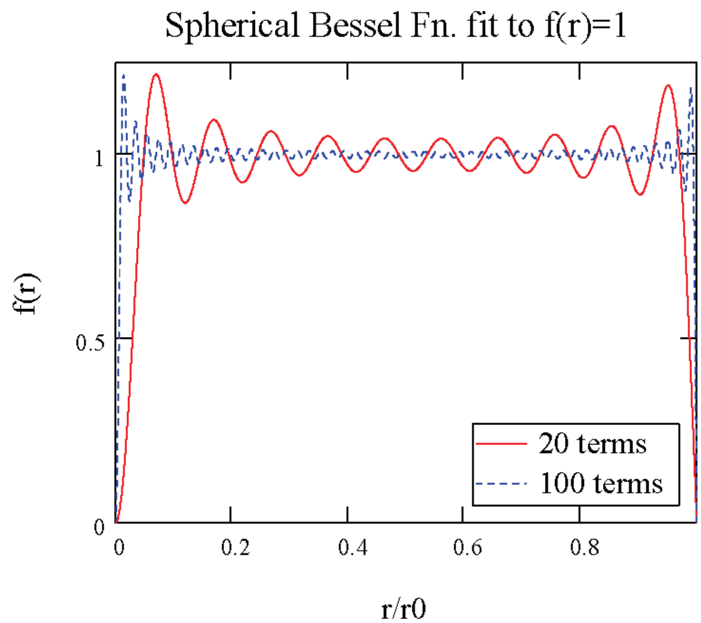


Figure 8-9 Spherical Bessel function fit to a distribution with a piecewise discontinuity.

9. Laplace equation

9.1 Origin of Laplace equation

Laplace's equation

$$\nabla^2 \Phi(\mathbf{r}) = 0 \tag{9.1}$$

occurs as the steady-state (time-independent) limit of a number of scalar second-order differential equations that span the range of physics problems. In electrostatics or Newtonian gravitation problems, the Φ field can be interpreted as defining a potential function (electrostatic or gravitational, respectively) in a source free region. The equation also occurs in thermodynamics, where Φ can be interpreted as the local temperature of a system in a steady state equilibrium.

To understand Laplace's equation, let's derive it in the context of Gauss's Law, which states that the net Electric flux crossing a closed boundary surface is proportional to the charge enclosed in the volume defined by the bounding surface:

$$\oint \mathbf{E} \cdot \hat{\mathbf{n}} dS = \frac{1}{\varepsilon_0} \int_{V \subset S} \rho dV = \frac{Q}{\varepsilon_0} \tag{9.2}$$

Where $\mathbf{E}$ is the electric field strength, $\hat{\mathbf{n}}$ is a unit normal to the surface S, and Q is the net charged enclosed in the region. The differential form of Gauss's law is given by Poisson's Equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{9.3}$$

where ρ is the charge density For a charge free region this reduces to

$$\nabla \cdot \mathbf{E} = 0 \tag{9.4}$$

For electrostatics, the electric field can be derived from a scalar potential function $\mathbf{E} = -\nabla\Phi$, which leads to Laplace's equation (9.1). Figure 9-1 shows a region of space for which Laplace's equation valid.

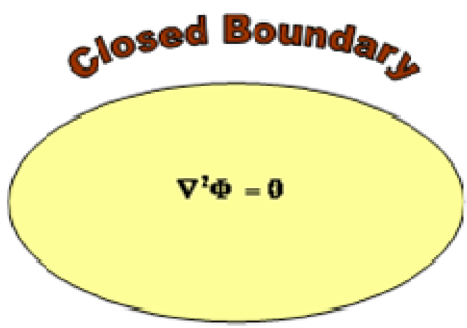


Figure 9-1. A closed region, in which Laplace's equation is valid

Laplace's equation has a unique (up to an overall constant value of the potential) solution for *either* of the following two sets of Boundary conditions:

- (Direchlet Boundary Conditions) The potential is defined everywhere on the boundary surface

$$\Phi_S = \Phi(\mathbf{r})\big|_S \tag{9.5}$$

OR

- (Neumann Boundary Conditions) The normal derivative of the potential is defined everywhere on the bounding surface:

$$\frac{\partial}{\partial_{\hat{n}}}\Phi\bigg|_S = -E_n(\mathbf{r})\big|_S = -E_S \tag{9.6}$$

9.2 Laplace equation in Cartesian coordinates

Laplace's equation in Cartesian coordinates can be written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi(x, y, z) = 0 \tag{9.7}$$

Solutions to this equation can be found by separation of variables in terms of product solutions: $X(x)Y(y)Z(z)$. The total solution can be expressed as a superposition over all of these "normal mode" solutions of the problem. For simplicity, lets limit the problem to a 2-dimensional space. Then, Laplace's equation reduces to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi(x, y) = 0 \tag{9.8}$$

By separation of variables the problem can be written in the form

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial x^2} = const \tag{9.9}$$

Which gives two sets of solutions

Case 1. X(x) is oscillatory, Y(y) is exponential

$$\begin{aligned} \frac{d^2}{dx^2}X(x) &= -k^2 X(x), \\ \frac{d^2}{dy^2}Y(y) &= +k^2 Y(y). \end{aligned} \tag{9.10}$$

The solutions to this case are

$$\begin{aligned} X(x) &= A\sin kx + B\cos kx, \\ Y(y) &= A\sinh ky + B\cosh ky. \end{aligned} \tag{9.11}$$

Case 2. Y(y) is oscillatory, X(x) is exponential

$$\begin{aligned} \frac{d^2}{dx^2}X(x) &= +k^2 X(x), \\ \frac{d^2}{dy^2}Y(y) &= -k^2 Y(y). \end{aligned} \tag{9.12}$$

This has solutions

$$\begin{aligned} X(x) &= A\sinh kx + B\cosh kx, \\ Y(y) &= A\sin ky + B\cos ky. \end{aligned} \tag{9.13}$$

To see how to apply these solutions, let's look at a rectangular box, shown in Figure 9-2, where the potential is known (that is, it has been measured) on each of the four surfaces. and $\nabla^2\Phi = 0$ everywhere inside the box.

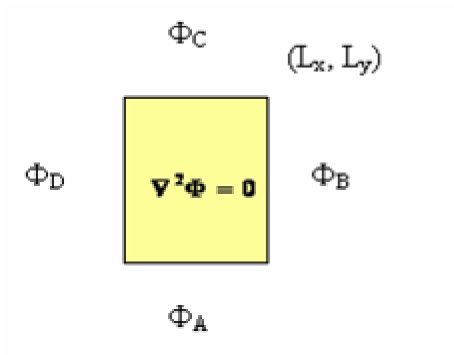


Figure 9-2 A rectangle of Length L_x and width L_y where the potential has been measured on all four surfaces.

By using the superposition principle, one can reduce this to four simpler problems, where the potential is non-zero on only one surface at a time, as see in Figure 9-3.

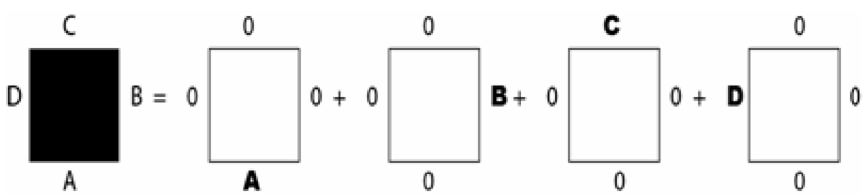


Figure 9-3 Superposition of four solutions to get a combined solution

The total solution can now be written as the superposition

$$\Phi(x,y)=\Phi_A(x,y)+\Phi_B(x,y)+\Phi_C(x,y)+\Phi_D(x,y) \qquad (9.14)$$

Let's examine solution for case A. The solutions for $X(x)$ must vanish at $[0,L_x]$ which can be satisfied by

$$X(x)=\sin(k_n x) \qquad (9.15)$$

where

$$k_m = \frac{m\pi}{L_x}. \tag{9.16}$$

Therefore $Y(y)$ must be a sum of $\sinh$ and $\cosh$ functions. The correct linear combination that vanishes at $y = L_y$ is given by

$$Y(y) = \sinh\left(k_n\left(L_y - y\right)\right) \tag{9.17}$$

Therefore,

$$\Phi_A(x, y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L_x} \sinh\frac{n\pi\left(L_y - y\right)}{L_x} \tag{9.18}$$

By a similar analysis the solutions for the remaining three surfaces can be found:

$$\Phi_C(x, y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L_x} \sinh\frac{n\pi y}{L_x}, \tag{9.19}$$

$$\Phi_B(x, y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{L_y} \sinh\frac{n\pi\left(L_x - x\right)}{L_y}, \tag{9.20}$$

$$\Phi_D(x, y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi y}{L_y} \sinh\frac{n\pi x}{L_y}. \tag{9.21}$$

❖ Solving for the coefficients

The solution to

$$\frac{d^2}{dx^2} X(x) = -k^2 X(x) \tag{9.22}$$

is a solution to a Sturm-Liouville differential equation, similar to that for the Fourier series expansion. The main difference is that the solution no longer satisfies periodic boundary conditions, but rather, Direchlet (or Neumann) boundary conditions at the end points of the interval $\left[0, L_x\right]$ The eigenfunctions are orthogonal on this interval, and satisfy the normalization condition

$$\int_0^{L_x} dx \sin\frac{n\pi x}{L_x} \sin\frac{m\pi x}{L_x} = \frac{L_x}{2}. \tag{9.23}$$

Using this relationship, one can then solve for the coefficients of the series expansion (9.18), giving

$$\begin{aligned} \int_0^{L_x} dx \sin\frac{m\pi x}{L_x} \Phi_A\left(x, y\right)\bigg|_{y=0} &= \int_0^{L_x} \sum_{n=1}^{\infty} \sin\frac{m\pi x}{L_x} A_n \sin\frac{n\pi x}{L_x} \sinh\frac{n\pi L_y}{L_x} \\ &= A_m \frac{L_x}{2} \sinh\frac{m\pi L_y}{L_x} \end{aligned} \tag{9.24}$$

or

$$A_n = \frac{2\int_0^{L_x} dx \sin\left(n\pi x / L_x\right) \Phi_A\left(x, 0\right)}{L_x \sinh\left(n\pi L_y / L_x\right)}. \tag{9.25}$$

Similarly, the results for the other three surfaces are given by

$$B_n = \frac{2\int_0^{L_x} dx \sin\left(n\pi x / L_y\right) \Phi_B\left(L_x, y\right)}{L_x \sinh\left(n\pi L_x / L_y\right)}, \tag{9.26}$$

$$C_n = \frac{2\int_0^{L_x} dx \sin\left(n\pi x / L_x\right) \Phi_C\left(x, L_y\right)}{L_x \sinh\left(n\pi L_y / L_x\right)}, \tag{9.27}$$

$$D_n = \frac{2\int_0^{L_x} dx \sin\left(n\pi x / L_y\right) \Phi_D\left(0, y\right)}{L_x \sinh\left(n\pi L_x / L_y\right)}. \tag{9.28}$$

Example: Consider a square 2-dimensional box of length L with sides have constant potentials

$$\Phi_A = \Phi_c = V_0 = -\Phi_B = -\Phi_D. \tag{9.29}$$

Find the potential inside the box.

In this case $L_x = L_y = L$, and the geometry is symmetric for reflections about the mid-plane wrt either the x or y directions. By symmetry, only the odd n terms survive

$$\begin{aligned} A_n = C_n = -B_n = -D_n &= \frac{2V_0 \int_0^{L_x} dx \sin\left(n\pi x / L\right)}{L \sinh\left(n\pi\right)} \\ &= \frac{2V_0}{n\pi} \int_0^{n\pi} dx' \sin\left(x'\right) = \frac{4V_0}{n\pi} \text{ for odd n,} \end{aligned} \tag{9.30}$$

$$\begin{aligned} \Phi\left(x, y\right) = &\frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{\sin \frac{n\pi x}{L} \left(\sinh \frac{n\pi y}{L} + \sinh \frac{n\pi(L-y)}{L} \right)}{\left(2n+1\right) \sinh\left(n\pi\right)} \\ &- \frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{\sin \frac{n\pi y}{L} \left(\sinh \frac{n\pi x}{L} + \sinh \frac{n\pi(L-x)}{L} \right)}{\left(2n+1\right) \sinh\left(n\pi\right)}. \end{aligned} \tag{9.31}$$

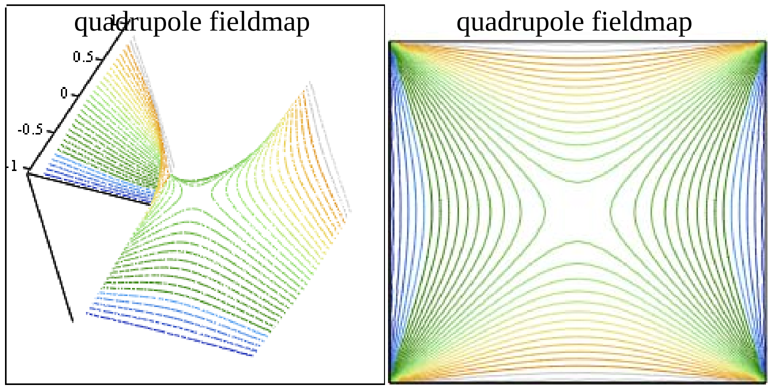


Figure 9-4 Field map of quadrupole potential surface

Figure 9-4 shows a field map of the potential surface. The series has difficulties fitting the results at the corners where the potential is discontinuous. Otherwise the result is consistent with what one might expect for a quadrupole field distribution.

Example: Solution in three dimensions for a rectangular volume with sides of length (a,b,c). Assume one surface (at $z=c$) is held at positive H.V. and the other 5 are grounded. Try a solution of the form

$$\Phi(x,y,z)=\sin(k_x x)\sin(k_y y)\sinh(k_z z). \tag{9.32}$$

This gives the eigenvalue equation

$$k_z^2=k_x^2+k_y^2. \tag{9.33}$$

The boundary conditions are

$$\Phi(x,y,z)\Big|_{x=0}^{x=a}=\Phi(x,y,z)\Big|_{y=0}^{y=b}=\Phi(x,y,z)\Big|_{z=0}=0. \tag{9.34}$$

This is satisfied by

$$k_x = \frac{m\pi}{a}; \quad k_y = \frac{n\pi}{b}; \quad k_z = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}. \qquad (9.35)$$

The sum over a complete set of states satisfying the boundary condition gives

$$\Phi(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm} \sin(m\pi x/a)\sin(n\pi y/b)\sinh(k_{mn} z). \quad (9.36)$$

Solving for the coefficients gives

$$A_{mn} = \frac{4}{ab\sinh k_{mn} c}\int_0^a dx \sin(m\pi x/a)\int_0^b dx \sin(m\pi y/b)\Phi_{z=c}(x,y), (9.37)$$

where

$$\mathrm{k}_{\mathrm{mn}} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}, \qquad (9.38)$$

$$A_{mn} = \begin{cases} \dfrac{16}{abmn\sinh k_{mn} c}, & \text{for } m,n \text{ odd}, \\ 0, & \text{otherwise.} \end{cases} \qquad (9.39)$$

9.3 Laplace equation in polar coordinates

Laplace's equation in 2-dimensional polar coordinates is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\Phi(r,\phi) = 0 \qquad (9.40)$$

The azimulthal coordinate is cyclic $\Phi(r,\phi+2n\pi) = \Phi(r,\phi)$. Try a product solution of the form

$$\Phi(r,\phi) = f_m(r)e^{im\phi}, \tag{9.41}$$

where $m = 0, \pm 1, \pm 2, \cdots$ The radial equation becomes

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right) f_m(r) = 0. \tag{9.42}$$

The operator does not mix powers of r, so the solutions are simple powers of r:

$$f_m(r) = r^{\pm m}. \tag{9.43}$$

However, for $m = 0$, this gives only one independent solution, the second solution is $\ln(r)$. The complete multipole series expansion can be written as

$$\Phi(r,\phi) = B_0 \ln(r/r_0) + \sum_{m=-\infty}^{m=\infty} \left(A_m \left(\frac{r}{r_0}\right)^m + B\left(\frac{r}{r_0}\right)^{-m} \right) e^{im\phi}, \tag{9.44}$$

where r_0 is some convenient scale parameter, used so that all the coefficients have the same dimensions.

9.4 Application to steady state temperature distribution

For steady-state temperature distributions the temperature T is a solution to Laplace's equation

$$\nabla^2 T(r,\phi) = 0. \tag{9.45}$$

Let us consider and infinitely long (OK, a very long) thick, cylindrical pipe, with inside radius a and outer radius b. Superheated water at $205^\circ C$ is flowing through the pipe which is buried underground at an ambient ground temperature of $55^\circ C$. Calculate the temperature differential along the radius of the pipe. Figure 9-5 shows a schematic cross section of the pipe.

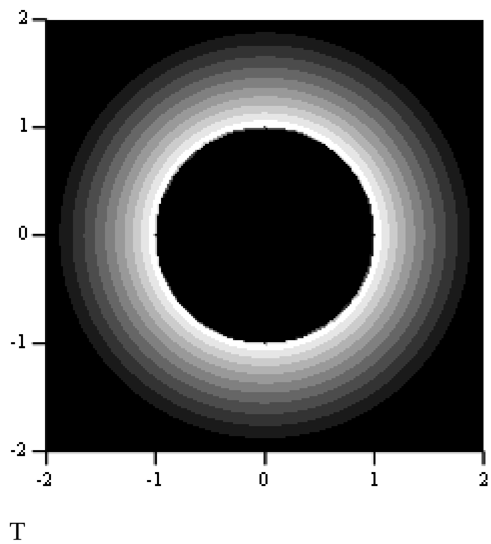


Figure 9-5 Temperature contour map of a cross section of a cylindrical pipe with superheated water flowing through it: The hotter regions of the pipe are whiter. Heat flow is radial, from hot to cold.

In this case, we have cylindrical symmetry. Therefore, there can not be any azimulthal dependence to the temperature distribution. The temperature can only depend on $m = 0$ terms. It can be written as

$$T(r) = A_0 + B_0 \ln(r/a) \tag{9.46}$$

Matching the temperature at the two boundaries gives

$$T(a) = 205^\circ C = A_0,$$
$$T(b) = 55^\circ C = A_0 + B_0 \ln(b/a), \tag{9.47}$$

which gives

$$A_0 = 205^\circ C \text{ and } B_0 = -150^\circ C / \ln(b/a). \tag{9.48}$$

9.5 The spherical capacitor, revisited

Consider a spherical capacitor, of radius r_0, consisting of two conducting hemispheres, one a positive high voltage, the other at negative high voltage. Pick the z-axis to be the symmetry axis. The potential distribution at the surface is given by

$$\Phi(r = r_0, \theta) = \begin{cases} +V_0 & \text{for } \cos\theta > 0, \\ -V_0 & \text{for } \cos\theta < 0 \end{cases}. \tag{9.49}$$

The solution is azimuthally symmetric, so it can be expanded in a Legendre series

$$\Phi_{in}(r,\theta) = V_0 \sum_{\text{odd } l}^{\infty} a_l \left(\frac{r}{r_0}\right)^l P_l(\cos\theta), \tag{9.50}$$

for the interior solution, or

$$\Phi_{out}(r,\theta) = V_0 \sum_{\text{odd } l}^{\infty} b_l \left(\frac{r_0}{r}\right)^{(l+1)} P_l(\cos\theta), \tag{9.51}$$

for the exterior solution. The solution is odd under reflection $(z \to -z)$; therefore, only terms odd in l survive. Note that the interior solution goes to zero at the origin, and the exterior solution goes to zero as $r \to \infty$. The potential must be continuous at the boundary $r = r_0$

$$\Phi_{in}(r_0, \theta) = \Phi_{out}(r_0, \theta), \tag{9.52}$$

implying

$$a_l = b_l. \tag{9.53}$$

Solving for the coefficients of a_l gives

$$\begin{aligned} a_l V_0 &= \frac{2l+1}{2} \int_{-1}^{1} V(x) P_l(x) dx \\ &= (2l+1) V_0 \int_0^1 P_l(x) dx \ \ (\text{for odd } l). \end{aligned} \tag{9.54}$$

The integral can be evaluated by use of the recursion formula

$$(2l+1) P_l = P'_{l+1} - P'_{l-1}, \tag{9.55}$$

giving

$$a_l = P_{l+1}(0) - P_{l-1}(0), \tag{9.56}$$

where

$$P_l(0) = \begin{cases} 0 & \text{for odd } l, \\ (-1)^{l/2} \dfrac{(l-1)!!}{l!!} & \text{for even } l. \end{cases} \tag{9.57}$$

Figure 9-6 shows the resulting contour map for the spherical capacitor. At the surfaces the potential goes to $\pm V_0$.asymptotically the distribution falls off as a dipole distribution

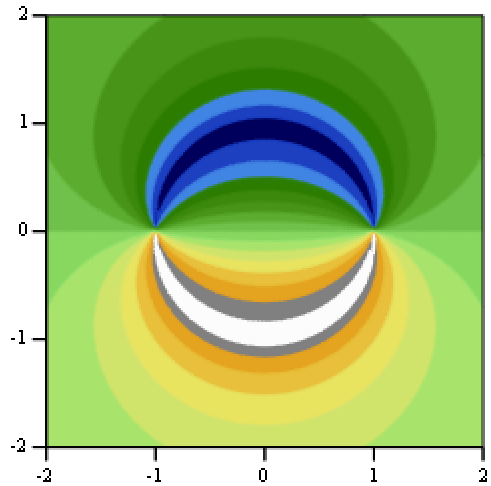


Figure 9-6 Potential contour map of the spherical capacitor in the taken in the (y, z) plane

❖ Charge distribution on a conducting surface

In the case of the spherical conductor, Laplace's equation is valid everywhere except at the conducting surface, the potential must come from a surface charge density on the conducting surfaces. When static equilibrium is reached, the potential within the thin conducting surfaces is a constant, so there cannot be any charge except at the surface layer. Moreover the Electric field must be normal to the surface or charge will continue to flow. Assuming a thin conducting layer gives the approximation

$$\rho(\mathbf{r}) = \left(\sigma_{in}(\theta) + \sigma_{out}(\theta)\right)\delta(r - r_0) \tag{9.58}$$

Integration over Poisson's equation in the radial direction then gives

$$\int_{r_0-\varepsilon}^{r_0+\varepsilon}\left(\frac{\partial E_r}{\partial r}\right)dr=\frac{1}{\varepsilon_0}\int_{r_0-\varepsilon}^{r_0+\varepsilon}2\sigma(\theta)\,\delta(r-r_0)dr$$

$$\begin{aligned}E_r\left(r_0+\varepsilon\right)-E_r\left(r_0+\varepsilon\right)&=\Delta E_r\left(r_0,\theta\right)\\&=\frac{\sigma_{out}(\theta)+\sigma_{in}(\theta)}{\varepsilon_0}=\frac{\sigma_{total}(\theta)}{\varepsilon_0}.\end{aligned}\tag{9.59}$$

where

$$\begin{aligned}E_r\Big|_{r=r_0}^{in}&=-\frac{\partial\Phi_{in}}{\partial r}\Big|_{r=r_0}=\frac{\sigma_{in}(\theta)}{\varepsilon_0},\\E_r\Big|_{r=r_0}^{out}&=-\frac{\partial\Phi_{out}}{\partial r}\Big|_{r=r_0}=\frac{\sigma_{out}(\theta)}{\varepsilon_0},\end{aligned}\tag{9.60}$$

This is a general result. For any conducting surface in static equilibrium, the field component normal to the surface is

$$E_n=\frac{\sigma}{\varepsilon_0},\tag{9.61}$$

which can easily be shown by constructing a infinitesimal Gaussian pillbox near the surface, with one side in the conductor and the other outside. The Electric field is discontinuous and points out of the surface wherever the density is positive, and into the surface, where it is negative. The surface charge density for the interior surface is given by.

$$\begin{aligned}-\frac{\partial\Phi_{in}}{\partial r}\Big|_{r=r_0}&=-\sum_{l=0}^{\infty}\frac{V_0a_{2l+1}(2l+1)}{r_0}\left(\frac{r}{r_0}\right)^{2l}P_{2l+1}(\cos\theta),\\\sigma_{in}(\theta)&=-\varepsilon_0\sum_{l=0}^{\infty}\frac{V_0a_{2l+1}(2l+1)}{r_0}P_{2l+1}(\cos\theta).\end{aligned}\tag{9.62}$$

Likewise, for the outer surface,

$$\sigma_{out}\left(\theta\right) = +\varepsilon_0 \sum_{l=0}^{\infty} \frac{V_0 a_{2l+1}\left(2l+2\right)}{r_0} P_{2l+1}\left(\cos\theta\right). \qquad (9.63)$$

9.6 Laplace equation with cylindrical boundary conditions

Laplace's equation in cylindrical coordinates is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right) V\left(r,\phi,z\right) = 0. \qquad (9.64)$$

Using separation of variables, one looks for product solutions of the form $V\left(r,\phi,z\right) = R\left(r\right)\Phi\left(\phi\right)Z\left(z\right)$. The function must satisfy periodic boundary conditions in the azimulthal coordinate, suggesting an expansion in Fourier series $\Phi\left(\phi\right) \sim e^{im\phi}$ should be tried. This gives rise to the eigenvalue equation

$$\frac{\partial^2}{\partial \phi^2} e^{im\phi} = -m^2 e^{im\phi} \quad \text{for } m = 0, \pm 1, \pm 2, \cdots. \qquad (9.65)$$

A similar expansion can be tried to separate the z dependence, giving rive to two possible sets of solutions

Case I:

$$\frac{\partial^2}{\partial z^2} e^{\pm kz} = k^2 e^{\pm ikz}, \quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m}{r^2}\frac{\partial^2}{\partial \phi^2} + k^2\right) R(r) = 0. \qquad (9.66)$$

Case II:

$$\frac{\partial^2}{\partial z^2}e^{\pm ikz} = -k^2 e^{\pm ikz}, \quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m}{r^2}\frac{\partial^2}{\partial \phi^2} - k^2\right)R(r) = 0. \quad (9.67)$$

This gives rise to the Bessel equation in the first instance and to the modified Bessel equation in the second instance. The complete solutions are built from product solutions of the form

$$\text{Case I: } V_I(r,\phi,z) \sim \begin{Bmatrix} J_m(kr) \\ N_m(kr) \end{Bmatrix} \begin{Bmatrix} \sinh(kz) \\ \cosh(kz) \end{Bmatrix} e^{im\phi} \quad (9.68)$$

and

$$\text{Case II: } V_I(r,\phi,z) \sim \begin{Bmatrix} I_m(kr) \\ K_m(kr) \end{Bmatrix} \begin{Bmatrix} \sin(kz) \\ \cos(kz) \end{Bmatrix} e^{im\phi}. \quad (9.69)$$

The choice of functions and allowed values of k are further restricted by the boundary conditions. Let us consider the case where one has Direchlet boundary conditions specified on the surface of a can, defined to be a cylinder of height L and of radius R. If we are interested on solving Laplace's equation in the interior of the can, then only the $J_m(kr)$ and $I_m(kr)$ Bessel functions can be used. The other radial functions are divergent at the origin. The solutions of Case I are appropriate if the potential is zero on the surface of the cylinder. Then the allowed values of k are restricted to fit the nodes of the Bessel function

$$J_m(kR) = 0 \quad (9.70)$$

or

$$k_{mn} = x_{mm} / R, \quad (9.71)$$

where x_{mm} are the zeros of the m^{th} Bessel function. The general solution to the first case is

$$V_I(r,\phi,z) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left(A_{mn}\frac{\sinh k_{mn}z}{\sinh k_{mn}L} + B_{mn}\frac{\sinh k_{mn}(L-z)}{\sinh k_{mn}L}\right)J_m\left(k_{mn}r\right)e^{im\phi}, \quad (9.72)$$

where the terms involving the A coefficients vanish on the surface $z=0$, and the terms involving the B coefficients vanish on the surface $z=L$. Both terms vanish at the cylindrical surface $r=R$. Notice that the z functions are pre-normalized to go to 1 on the non-vanishing surface. This is a common technique. Let $V_{IA}(r,\phi,L)$ be the potential on the surface $z=L$.

Then, by integration,

$$\begin{aligned}
&\int_{-\pi}^{\pi}d\phi\int_0^1 xdxV_{IA}(r,\phi,L)J_m\left(x_{mn}r/R\right)e^{-im\phi} \\
&= \sum_{m'n'}A_{mn}\int_{-\pi}^{\pi}d\phi\int_0^1 xdxJ_m\left(x_{mn}r/R\right)J_{m'}\left(x_{m'n'}r/R\right)e^{-im\phi}e^{im'\phi} \\
&= A_{mn}\left(2\pi\right)\left(\frac{J_{m+1}^2\left(x_{mn}\right)}{2}\right) = \pi J_{m+1}^2\left(x_{mn}\right)A_{mn}
\end{aligned} \quad (9.73)$$

or

$$A_{mn} = \frac{1}{\pi J_{m+1}^2\left(x_{mn}\right)}\int_{-\pi}^{\pi}d\phi\int_0^1 xdxV_{IA}(r,\phi,L)J_m\left(x_{mn}r/R\right)e^{-im\phi}. \quad (9.74)$$

Likewise for the surface at $z=0$:

$$B_{mn} = \frac{1}{\pi J_{m+1}^2\left(x_{mn}\right)}\int_{-\pi}^{\pi}d\phi\int_0^1 xdxV_{IB}(r,\phi,0)J_m\left(x_{mn}r/R\right)e^{-im\phi}. \quad (9.75)$$

The remaining surface at $r = R$ is a solution of the modified Bessel equation, where the nodes of $Z(z)$ vanish at the end points of the interval $[0, L]$:

$$V_{II}(r,\phi,z) = \sum_{m=-\infty}^{\infty} C_{mn} \sin(k_{mn}z) \frac{I_m(k_{mn}r)}{I_m(k_{mn}R)} e^{im\phi}, \tag{9.76}$$

where

$$k_{mn}L = n\pi \tag{9.77}$$

and $$V_{II}(r,\phi,z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin\left(\frac{n\pi z}{L}\right) \frac{I_m\left(\frac{n\pi r}{L}\right)}{I_m\left(\frac{n\pi R}{L}\right)} e^{im\phi}. \tag{9.78}$$

Solving for the boundary conditions at surface C gives

$$V_{IIC}(R,\phi,z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin\left(\frac{n\pi z}{L}\right) e^{im\phi}. \tag{9.79}$$

Integrating

$$\begin{aligned}
&\int_{-\pi}^{\pi} d\phi \int_{o}^{L} dz V_{IIC}(R,\phi,z) e^{-im\phi} \sin\left(\frac{n\pi z}{L}\right) \\
&= \sum_{m'=-\infty}^{\infty} \sum_{n'=1}^{\infty} \int_{-\pi}^{\pi} d\phi \int_{o}^{L} dz C_{m'n'} \sin\left(\frac{n'\pi z}{L}\right) e^{im'\phi} e^{-im\phi} \sin\left(\frac{n\pi z}{L}\right) \\
&= C_{mn}(2\pi)\left(\frac{L}{2}\right) = \pi L C_{mn}
\end{aligned} \tag{9.80}$$

or

$$C_{mn} = \frac{1}{\pi L} \int_{-\pi}^{\pi} d\phi \int_{o}^{L} dz V_{IIC}(R,\phi,z) e^{-im\phi} \sin\left(\frac{n\pi z}{L}\right). \tag{9.81}$$

The total solution is a superposition of the above three solutions:

$$V(r,\phi,z) = V_{IA}(r,\phi,z) + V_{IB}(r,\phi,z) + V_{IIc}(r,\phi,z)$$
$$= \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left(A_{mn}\frac{\sinh k_{mn}z}{\sinh k_{mn}L} + B_{mn}\frac{\sinh k_{mn}(L-z)}{\sinh k_{mn}L}\right)J_m\left(k_{mn}r\right)e^{im\phi} \quad (9.82)$$
$$+\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin\left(\frac{n\pi z}{L}\right)\frac{I_m\left(\frac{n\pi r}{L}\right)}{I_m\left(\frac{n\pi R}{L}\right)}e^{im\phi}.$$

❖ Solution for a clyindrical capacitor

Consider a metal can with three metallic surfaces, held at three different potentials. For simplicity, let the top and bottom surfaces be held at positive and negative high voltages $\pm V_0$, respectively; Let the cylindrical side be grounded:

$$V_{IA} = -V_{IB} = V_0 \quad \text{and} \quad V_{IIC} = 0 \qquad (9.83)$$

By cylindrical symmetry, the sum over m vanishes, except for $m = 0$. The coefficients to be determined are

$$A_{0n} = -B_{0n} = \frac{2V_0}{J_1^2\left(x_{0n}\right)}\int_0^1 x dx J_0\left(x_{0n}r/R\right). \qquad (9.84)$$

where all the other coefficients vanish due to the boundary conditions. This integral can be solved by use of the recursion formula

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{9.85}$$

Letting $p = 1$ gives

$$\int_0^1 x dx_0 J(ax) = \frac{1}{a^2}\int_0^a x' dx' J_0(x') = \frac{1}{a^2} x' J_1(x')\Big|_0^a = \frac{J_1(a)}{a}, \tag{9.86}$$

Leading to the result

$$A_{0n} = -B_{0n} = \frac{2V_0}{x_{0n} J_1(x_{0n})}. \tag{9.87}$$

Putting it all together, the potential everywhere inside the can is given by

$$V(r,z) = \sum_{n=1}^{\infty} \frac{2V_0}{x_{0n} J_1(x_{0n})}\left(\frac{\sinh k_{0n} z}{\sinh k_{0n} L} - \frac{\sinh k_{0n}(L-z)}{\sinh k_{0n} L}\right) J_0(k_{0n} r), \tag{9.88}$$

where $k_{on} = x_{0n} / R$.

10. Time dependent differential equations

Time changes all things. It is responsible for evolution at the biological and cosmological scales. Time makes motion possible. It is the apparent casual behavior of events that allows us to make sense of our universe. Newton considered time to flow uniformly for all observers, a scalar parameter against which our lives are played out. Special relativity showed that space and time are geometrically related and transform like vectors in Minkowski space. But there is an arrow of time, nonetheless. There is no continuous Lorentz transform that takes a time-like vector with a positive time direction and converts it to one with a negative time sense. Thermodynamic processes are subject to the laws of entropy, which may signal the eventual heat death of our universe. More importantly for our purposes, the motions of classical particles are well behaved single-valued functions of time. Given a complete set of initial conditions and an adequate theoretical framework, we can project the past into the future and make useful predictions about outcomes. The solution of the initial value problem forms the core of dynamics.

10.1 Classification of partial differential equations

Laplace's equation

$$\nabla^2\Psi(\mathbf{r}) = 0 \tag{10.1}$$

is an example of an *elliptic differential equation*, so-called because the differential operator takes on a elliptic form $D_x^2 + D_y^2 + D_z^2$. Such equations have solutions if the function or its derivative is defined on a closed, bounding surface. Adding a potential term to the operator does not change the character of the solution. For example, the equation,

$$\left(\nabla^2 + K(\mathbf{r})\right)\Psi(\mathbf{r}) = 0 \tag{10.2}$$

is also classified as an elliptic differential equation, and the equation has a unique, stable solution if it satisfies Direchlet or Neumann boundary conditions.

We are used to thinking of time as an additional dimension, but it is a peculiar one. Solutions for time dependent problems are defined in terms of specifying a set of initial conditions, If one considers time as a fourth coordinate, then the initial value problem is equivalent to a boundary value problem, where the appropriate boundary conditions are to be specified over an open hyper-surface, usually defined at a constant time, $t = t_0$. Mathematically, the character of the differential operator differs from the elliptic character of Laplace's equation.

For example, the diffusion equation

$$\left(\nabla^2 - \frac{1}{\alpha^2}\frac{\partial}{\partial t}\right)\Psi(\mathbf{r},t) = 0 \tag{10.3}$$

is linear in time and the differential operator has a parabolic signature $\sum_i D_i^2 - \alpha^{-2} D_t$. It is an example of a *parabolic differential equation*. Analysis shows that stable unique solutions for one direction in time can be found using either Direchlet or Neumann boundary conditions on a open surface. The arrow of time is forward, and thermodynamic systems flow in the direction of increasing entropy.

The wave equation

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial^2 t}\right)\Psi(\mathbf{r},t) = 0 \tag{10.4}$$

is another common time-dependent partial differential equation. It is second order in time, but its time signature has the opposite sign from the Laplacian operator: $\sum_i D_i^2 - c^{-2} D_t^2$. This is an example of a *hyperbolic differential equation*. The wave equation has stable solutions in either time direction, but because it is second order in time, it satisfies Cauchy boundary conditions on an open surface. Cauchy boundary conditions require that both the function and its normal derivative be specified at some initial or final time $t = t_0$. Finally, the Schrödinger equation

$$\left(\frac{-\hbar^2}{2m}\nabla^2 - i\hbar\frac{\partial}{\partial t}\right)\Psi(\mathbf{r},t) = 0 \tag{10.5}$$

is first order in time. Like the diffusion equation, it satisfies Direchlet or Neumann boundary conditions on an open surface.

Because of the imaginary i in the definition of the time operator, the operator is Hermitian, and the equation has sable solutions in either time direction. Table 10-1 lists some common differential equations and their boundary conditions.

Removal of the time dependence in any of the above equations leads to the Helmholtz equation, which has an elliptic character. Therefore, to solve these equations completely, one must specify not only the functions and/or their derivatives throughout the volume at some initial time, but also specify their behavior at some bounding surface for all time. If, however, the behavior at the boundary is static in character, then the problem can be separated into two problems:

- the behavior at the static boundary can be fitted to a general solution to the time-independent equation, ignoring the time behavior of this part of the problem, (this usually results in Laplace's equation), and
- a particular solution to the time-dependent problem can be added to this which satisfies the trivial boundary condition that either the function or its normal derivative vanish at the bounding surface.

The total solution is then

$$\Psi(\mathbf{r},t)=\Psi_{static}(\mathbf{r})+\Psi_{particular}(\mathbf{r},t), \tag{10.6}$$

where $\Psi_{static}(\mathbf{r})$ is given the job of satisfying any non-trivial, but static, boundary conditions at the enclosing surface.

Table 10-1 A list of common partial differential equations and their allowed boundary conditions

Character	Equation	Boundary Conditions
Elliptic	Laplace and Helmholtz Equations	Direchlet or Neumann on a closed surface.
Hyperbolic	Wave Equation	Cauchy on an open surface
Parabolic	Diffusion Equation	Direchlet or Neumann on an open surface. (stable in one direction)
Complex Parabolic	Schrödinger Equation	Direchlet or Neumann on an open surface.

The usual procedure is to first solve for the steady state background term, and subtract its contribution from the initial condition of the function in the interior volume. The remaining time dependent problem can then be solved by separation of variables, in terms of product solutions

$$\Psi_{particular(k)}\left(\mathbf{r},t\right)=\Phi_{k}\left(\mathbf{r}\right)T_{k}\left(t\right), \qquad (10.7)$$

where $\Phi_k\left(\mathbf{r}\right)$ are the stationary normal nodes of the space problem. These normal modes are solutions to the Helmholtz equation

$$\nabla^2\Phi(\mathbf{r}) = -k^2\Phi(\mathbf{r}), \tag{10.8}$$

in the absence of any complicating additional potential term.

10.2 Diffusion equation

The diffusion equation is often used to model stochastic heat flow. It is valid where the thermal resistance is sufficient, and time scales long enough, to allow definition of a local temperature in a thermodynamic medium. It can be derived from two basic assumptions

- The gradient of the temperature T is proportional to the heat flux $\mathbf{Q} \propto \nabla T$.
- the divergence of the heat flux is proportional to the rate of change of temperature $\nabla \cdot \mathbf{Q} \propto \partial T / \partial t$.

Colloquially, the first equation states that heat flows from hot to cold, while the second states that temperature changes fastest where the divergence is greatest. When the temperature reaches a steady state condition one gets Laplace's equation, which has zero divergence:

$$\nabla \cdot \mathbf{Q} = 0 \Rightarrow \nabla^2 T = 0. \tag{10.9}$$

In the general case, before steady state equilibrium has been reached, the two assumptions give rise to the diffusion equation

$$\nabla^2 T = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}, \tag{10.10}$$

where α^2 is a property of the material that is proportional to the thermal conductivity.

The time eigenstates of this equation are given by

$$T(t) = e^{\pm(k\alpha)^2 t}. \tag{10.11}$$

The negative sign is chosen, since one expects the system to relax to a steady state temperature distribution, given sufficient time. The terms with positive signs represent the time reversed problem, which is unstable, since the terms exponentially diverge. The boundary values to the time independent Helmholtz equation, (10.8), restrict the possible values of k, which in turn restrict the $1/e$ decay times of the normal modes

$$t_k = 1/(k\alpha)^2. \tag{10.12}$$

The total solution can be written as

$$T(\mathbf{r},t) = T_{steadyState}(\mathbf{r}) + \sum_k A_k \Phi_k(\mathbf{r}) e^{-t/t_k}. \tag{10.13}$$

Note that the modes with larger values of k decay faster (since they have smaller time constants), and that

$$\lim_{t\to\infty} T(\mathbf{r},t) = T_{steadyState}(\mathbf{r}). \tag{10.14}$$

Note as well that the initial value of the particular solution to the time dependent problem is not given by $T(\mathbf{r},0)$, but is given instead by the difference

$$T_{particular}(\mathbf{r},t) = T(\mathbf{r},t) - T_{steadyState}(\mathbf{r}), \tag{10.15}$$

evaluated in the limit as $t \to 0$. The coefficients A_k are determined by solving the initial value problem

$$T_{particular}(\mathbf{r},0) = T(\mathbf{r},0) - T_{steadyState}(\mathbf{r}) = \sum_k A_k \Phi_k(\mathbf{r}). \qquad (10.16)$$

Example: Heat flow in a bar

Consider a long, thin iron bar that is insulated along its length, but not at its ends. Originally the bar is in thermal equilibrium at room temperature, $22^\circ C$, but at time $t = 0$, one end is inserted into a vat of ice water at $0^0 C$. Calculate the temperature distribution in the bar as a function of time and find its final steady state temperature distribution.

Since the bar is thin and its sides insulated, this can be treated as a problem in one space dimension x.

The initial condition is given by the uniform temperature distribution,

$$T(x,0) = 22^\circ C. \qquad (10.17)$$

The steady state condition, treating the room and the vat as infinite heat sinks, gives the static boundary conditions,

$$T(0,t) = 22^\circ C \text{ and } T(L,t) = 0. \qquad (10.18)$$

The steady state problem is a solution to Laplace's equation in one-dimension

$$\frac{d^2T(x)}{dx^2} = 0, \qquad (10.19)$$

which has the solution

$$T_{ss}(x) = T_0 - \Delta T \frac{x}{L}, \tag{10.20}$$

where $T_0 = \Delta T = 22^\circ C$. Therefore, the steady state limit corresponds to a uniform temperature drop from the hot face to the cold face of the bar.

The initial value problem for the particular time-dependent solution is given by

$$T_p(x,0) = T_0 - T_{ss}(x) = \Delta T \frac{x}{L}. \tag{10.21}$$

This excess temperature component decays in time, and the system relaxes to its steady-state limit. The normal modes of the time-dependent problem as sine functions that go to zero at the end points of the interval $[0, L]$. Therefore the product solutions take the form

$$\Phi_k(x) T_k(t) = \sin(kx) e^{-t/t_k}, \tag{10.22}$$

where

$$k = n\pi / L \quad \text{and} \quad t_k = (L / n\pi\alpha)^2. \tag{10.23}$$

The solution to the initial value problem is

$$T_p(x,0) = T_0 \sum_n A_n \sin \frac{n\pi x}{L}, \tag{10.24}$$

with coefficients given by

$$A_n = \frac{2}{L}\int_0^L dx \frac{T_p(x,0)}{T_0}\sin\frac{n\pi x}{L} = \frac{2}{L}\int_0^L dx \frac{x}{L}\sin\frac{n\pi x}{L}. \qquad (10.25)$$

The total solution summing the steady state and time dependent contributions is

$$T(x,t) = T_0\left(1 - \frac{x}{L} + \sum_n A_n \sin\frac{n\pi x}{L} e^{-(n\pi\alpha)^2 t/L^2}\right). \qquad (10.26)$$

The decay times fall as $\sim 1/n^2$, so after sufficient time has passed only the first few modes are of importance. In the time-reversed problem the opposite situation arises, the large n components would grow exponentially as one goes further back into the past. The solution of the time reverse problem depends sensitively on the initial conditions, one must be able to bound very small high frequency components to impossibly small constraints, and the results are therefore unstable under small perturbations. The diffusion equation can be reliably used only to predict the future behavior of a thermodynamic system.

10.3 Wave equation

Material waves are time-dependent fluctuations in a medium that transport energy and momentum to the boundaries of the medium. They have a characteristic velocity of propagation that is a property of the specific medium. Maxwell showed that a self-consistent solution of the equations of electricity and magnetism, then though of as disparate, but interacting, fields required that the electric and magnetic fields simultaneously satisfy a wave equation where the wave velocity is determined by the speed of light. From that he deduced that the origin of light is fundamentally electromagnetic in character. Marconi later confirmed this hypothesis with the discovery of radio waves, induced by the oscillation of electric charges in an antenna. Before then, the Michelson-Morley experiment had already demonstrated that the speed of light in free space was independent of the properties of a underlying medium, referred to as the either. Today we are comfortable with the notion that the electromagnetic field is an intrinsic property of spacetime and does not require an underlying medium for its propagation.

Waves are classified as to whether the amplitude of oscillation is along (longitudinal) or transverse to the direction of propagation. Vibrating strings and waves on the surface of a pond are examples of transverse waves, while sound in a gaseous medium is a purely longitudinal disturbance, since gases cannot support a shear force. Waves in solids are more complex, having both transverse and longitudinal modes, usually with different velocities of propagation. In most cases, the linear character of the wave equation is the result a small amplitude approximation to a more complex non-linear theory, one which includes dissipative and dispersive contributions.

In its simplest form, the wave equation relates the second-order space and time derivatives of some fluctuation, to the wave velocity v:

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial^2 t}\right)\Psi(\mathbf{r},t) = 0. \tag{10.27}$$

In the case of a string or a surface wave, this fluctuation is a transverse displacement. In the case of sound vibrations in a medium it represents the propagation of a pressure disturbance. The stored energy density of the wave is proportional to the square of this amplitude. Solution of the wave equation often involves solving for the normal modes of oscillation in time via separation of variables. This involves separating the wave into it frequency components in the time domain:

$$\Psi(\mathbf{r},t) = \sum_{\omega} \Phi_{\pm\omega}(\mathbf{r}) e^{\pm i\omega t} = \sum_{k=\omega/c} A_k(\mathbf{r}) \cos \omega t + B_k(\mathbf{r}) \sin \omega t, \tag{10.28}$$

where the wave number $k = \omega / c$ is often restricted to discrete values by the boundary conditions at the bounding surface of the medium. For fixed frequency, the normal modes of oscillation, which can be denoted as $\Phi_k(\mathbf{r})$, are solutions to the time independent Helmholtz equation (10.8). Note that there are two initial conditions that must be satisfied. At time $t = 0$, one must specify both the initial function and its time derivative, i.e.,

$$\Psi(\mathbf{r},0) = \sum_{k=\omega/c} A_k(\mathbf{r}) \cos \omega t \tag{10.29}$$

and

$$\Psi'(\mathbf{r},0) = \sum_{k=\omega/c} \omega B_k(\mathbf{r}) \sin \omega t, \tag{10.30}$$

where

$$\Psi'(\mathbf{r},0) = \left(\partial\Psi(\mathbf{r},t)/\partial t\right)_{t=0}. \tag{10.31}$$

❖ Pressure waves: standing waves in a pipe

Sound waves in a gaseous medium are longitudinal waves. At a closed rigid boundary, the longitudinal displacement of the medium goes to zero, and one has a displacement node that the boundary. Correspondingly, the pressure at such a boundary is a maximum or minimum and therefore the pressure has an anti-node that the boundary. Stated in other terms, the pressure at a closed boundary satisfies Neumann boundary conditions

$$\left.\frac{\partial P(x,t)}{\partial t}\right|_{\text{closed boundary}} = 0. \tag{10.32}$$

At an open surface, there is no impedance and the pressure differential across the boundary drops to zero. Therefore a stationary wave would satisfy Direchlet boundary conditions at an open boundary.

$$P(x,t)\big|_{\text{open boundary}} = 0. \tag{10.33}$$

If one applies this to an organ pipe of length L with a open end at $x=0$ and a closed end at $x=L$, the allowed standing wave nodes are

$$P(x,t) \propto \sin(kx)\left[A\cos\left(\omega t+\phi\right)\right], \tag{10.34}$$

where ϕ is a phase angle given by the initial conditions, and $P(x,t)$ is a stationary solution to the wave equation

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2}{\partial^2 t}\right)P\left(x,t\right)=0. \tag{10.35}$$

The wave velocity in a gas is given by $v=\sqrt{Y/\rho}$ where Y is Young's modulus (one-third of the bulk modulus) and ρ is the density. The boundary conditions for a half-open pipe require

$$kL=\left(n+\tfrac{1}{2}\right)\pi, \tag{10.36}$$

with an angular frequency given by

$$\omega=kv\,. \tag{10.37}$$

Usually a organ pipe is sounded to emphasize a nearly pure harmonic note at the fundamental frequency, corresponding to $n=0$.

❖ The struck string

The struck string on a string instrument satisfies Direchlet boundary conditions at its end points

$$y(x,t)\Big|_{x=0}^{x=L}=0. \tag{10.38}$$

where y is the transverse displacement of the string from its equilibrium position. Its normal modes of motion are given by

$$\Phi_n(x) = \sin\frac{n\pi x}{L}, \tag{10.39}$$

where $k = n\pi / L = \omega / v$. The wave velocity is given by $v = \sqrt{T/\mu}$ where T is the tension and μ is the mass per unit length. The general solution can be written as

$$y(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi vt}{L} + B_n \sin\frac{n\pi vt}{L}\right)\sin\frac{n\pi x}{L}. \tag{10.40}$$

The string has a fundamental harmonic for $\omega/2\pi = nv/2L$ with a rich texture of harmonics depending on how the string is struck. The actual sound produced by a stringed instrument is significantly modified by its sound board, but let's analyze the response of the string in isolation. The initial conditions are given by

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \tag{10.41}$$

and by

$$y'(x,0) = \sum_{n=1}^{\infty} B_n \omega_n \sin\frac{n\pi x}{L} \tag{10.42}$$

where $\omega_n = n\pi v / L$.

The solution for the coefficients are given by

$$A_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L} y(x,0)dx \tag{10.43}$$

and

$$B_n = \frac{2}{\omega_n L}\int_0^L \sin\frac{n\pi x}{L} y'(x,0)dx. \qquad (10.44)$$

As an example, suppose the string is struck at its exact middle by an impulsive force. Then the initial conditions can be expressed approximately by a delta-function contribution to the instantaneous velocity distribution at the initial time $t = 0$:

$$y(x,0) = 0; \quad y'(x,0) = \lambda_0 \delta(x - L/2) \qquad (10.45)$$

Therefore, $A_n = 0$ and

$$B_n = \frac{2}{\omega_n L}\int_0^L \sin\frac{n\pi x}{L}\delta(x - L/2)dx = \frac{2L}{n\pi v}\sin(n\pi/2). \qquad (10.46)$$

Only terms odd in n contribute, with the time evolution of the original delta function given by

$$y(x,t) = \sum_{n=0}^{\infty}\left(\frac{(-1)^n 2L}{(2n+1)\pi v}\right)\sin\frac{(2n+1)\pi x}{L}\sin\frac{(2n+1)\pi vt}{L}. \qquad (10.47)$$

❖ The normal modes of a vibrating drum head

A circular drum head can be approximated as a vibrating membrane, clamped at its maximum radius r_0. The amplitude of transverse motion in the z direction is a solution to the wave equation

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial^2 t}\right)Z(\mathbf{r},t) = 0 \qquad (10.48)$$

The problem separates in polar coordinates, giving normal stationary modes that can be written in terms of cylindrical Bessel functions.

$$Z_{mn}(\mathbf{r}) = J_m(a_{mn} r / r_0) e^{im\phi} \tag{10.49}$$

with a total solution given by

$$Z_{mn}(\mathbf{r}) = \sum_{mn} (A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t) J_m(a_{mn} r / r_0) e^{im\phi} \tag{10.50}$$

Where the normalization condition

$$\begin{aligned} &\int_0^1 x dx_{m'} J(a_{m'n'} x) J_m(a_{mn} x) \\ &\times \int_{-\pi}^{\pi} d\phi e^{-im'\phi} e^{im\phi} = \pi J_{m+1}^2(a_{mn}) \delta_{mm'} \delta_{nn'} \end{aligned} \tag{10.51}$$

can be used to determine the coefficients.

The allowed wave numbers are those given by the zeros of the Bessel functions

$$k_{mn} = a_{mn} r / r_0, \quad J_m(a_{mn}) = 0 \tag{10.52}$$

Since these are transcendental numbers the vibration frequencies are not simple harmonic multiples of each other, therefore the sound made by a percussion instrument, such a circular drum, often sounds discordant, with frequencies given by

$$f_{mn} = \omega_{mn} / 2\pi = a_{mn} v / 2\pi r_0 . \tag{10.53}$$

The first few normal modes of the vibrating membrane are shown in Figure 10-1 and Figure 10-2.

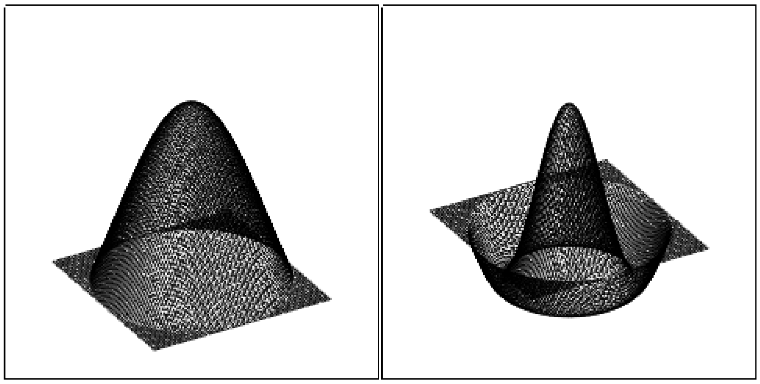


Figure 10-1 The first two nodes of the m=0 Bessel function

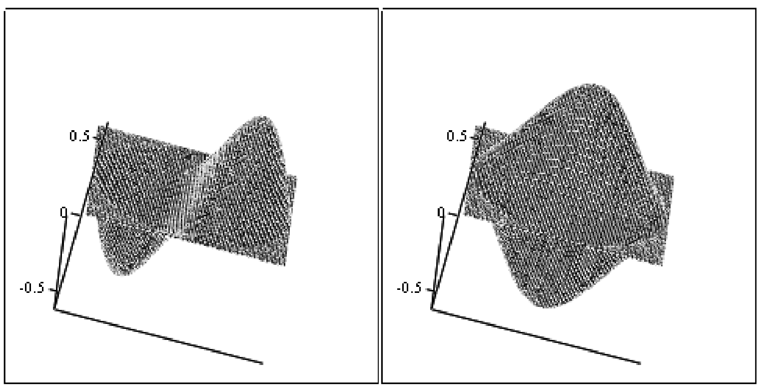


Figure 10-2 The first node of the m=1 Bessel function has two orientations corresponding to $\sin\phi$ **and** $\cos\phi$ **solutions**

Discussion Problem: Solve for the time evolution of a circular drum head struck impulsively at its exact middle by a drum stick. The initial conditions are

$$Z(r,\phi,0)=0;\quad Z'(r,\phi,0)=\lambda_0\delta(x)\delta(y). \tag{10.54}$$

Use the Jacobean of transformation from polar to Cartesian coordinates to carry out the integrals for the coefficients

$$\int rdrd\phi f(\mathbf{r}) = \int dxdyf(\mathbf{r})\delta(x)\delta(y) = 2\pi f(0). \tag{10.55}$$

Note that only $m = 0$ terms contribute to the result.

10.4 Schrödinger equation

The Schrödinger equation is given by

$$H\Psi(\mathbf{r},t) = i\hbar \frac{\partial \Psi(\mathbf{r},t)}{\partial t}, \tag{10.56}$$

where H is the Hamiltonian operator and $|\Psi(\mathbf{r},t)|^2$ represents the probability density of finding a particle at a given location in space. Therefore the equation represents the evolution of the probability amplitude in time. If H is a Hermitian operator, the probability is conserved and a single particle state is assigned a total unit probability of being located somewhere in space

$$\int d^3r \, |\Psi(\mathbf{r},t)|^2 = 1. \tag{10.57}$$

The equation is first order in time, like the diffusion equation. Unlike the diffusion equation the time behavior is oscillatory, therefore the time evolutions is well-behaved for propagation into past or future time. Separation of variables gives product solutions of the form

$$\Psi_k(\mathbf{r},t) = \Psi_k(\mathbf{r})e^{-i\omega t}, \tag{10.58}$$

where

$$H\Psi_k(\mathbf{r}) = E_k\Psi_k(\mathbf{r}) = \hbar\omega_k\Psi_k(\mathbf{r}) = (\hbar k)^2/2m \tag{10.59}$$

and k is a solution to the eigenvalue equation for the stationary modes of motion.

10.5 Examples with spherical boundary conditions

❖ Quantum mechanics in a spherical bag

The Time-Independent Schrödinger equation for a freely moving particle, in the absence of a potential, is given by

$$H = \frac{\mathbf{p}^2}{2m} = -\frac{\hbar^2}{2m^2}\nabla^2. \tag{10.60}$$

This can be rewritten as the Helmholtz equation

$$\nabla^2\Psi(\mathbf{r}) = -k^2\Psi(\mathbf{r}), \tag{10.61}$$

where $E = \frac{(\hbar k)^2}{2m}$.

If the particle is put into a infinite well of radius $r = r_0$, the wave function vanishes at the spherical boundary. The product solutions can then be written as

$$\Psi_{lm,n}(\mathbf{r}) = \sqrt{\frac{2}{r_0^3 j_{l+1}^2(a_{l,n})}}\, j_l(a_{l,n} r / r_0) Y_{lm}(\theta,\phi), \tag{10.62}$$

where the normalization is chosen so that

$$\int_0^{r_0} r^2 dr \oint d\Omega \Psi_{lm,n}(\mathbf{r})\Psi^*_{lm,n}(\mathbf{r}) = 1. \tag{10.63}$$

The allowed eigenvalues for the energies are constrained by the boundary conditions to the discrete set

$$E_{nl} = \hbar\omega_{nl} = \frac{\left(\hbar a_{ln}/r_0\right)^2}{2m}. \tag{10.64}$$

The energy does not depend on the m state value, so the energies are $(2l+1)$-fold degenerate for any given l value. In general the total wave function need not be in an eigenstate of energy, so the wave function at some initial time can be written as a sum over all possible states

$$\Psi(\mathbf{r}) = \sum_{nlm} c_{nlm}\Psi_{nlm}(\mathbf{r}), \tag{10.65}$$

where $|c_{nlm}|^2$ denotes the fractional probability that it is any given state.

The time evolution of this wave packet is given by

$$\Psi(\mathbf{r},t) = \sum_{nlm} c_{nlm}\Psi_{nlm}(\mathbf{r})e^{-i\omega_{nl}t}. \tag{10.66}$$

If a particle where known to be localized at some point within the sphere at a fixed time, the different time behaviors of normal modes would cause its position probability to disperse in time.

❖ Heat flow in a sphere

Consider a sphere heated to a uniform temperature T_0 at some initial time t_0, then immediately dropped into a quenching bath at a temperature T_f. Calculate it temperature distribution at later times.

The temperature distribution satisfies the initial condition

$$T(\mathbf{r},t_0)=T_0 \tag{10.67}$$

and must satisfy the boundary condition

$$T(\mathbf{r},t)\Big|_{r=r_0}=T_f \quad \text{for } t>t_0. \tag{10.68}$$

Therefore, it can be expanded in the series solution

$$T(\mathbf{r},t)=T_f+\sum_{lm}A_{lmn}Y_{lm}(\theta,\phi)\sum_{n=1}^{\infty}j_l\left(a_{ln}r/r_0\right)e^{-(t-t_0)/t_{lmn}}, \tag{10.69}$$

where T_f is the steady-state solution. By spherical symmetry, only $l=m=0$ terms contribute, and the time constants are given by $t_{lmn}=\left(\alpha a_{ln}/r_0\right)^{-2}$. Therefore the solution can be written as

$$T(\mathbf{r},t)-T_f=\sum_{n=1}^{\infty}A_n j_0\left(a_{0n}r/r_0\right)e^{-(t-t_0)/t_{lmn}}, \tag{10.70}$$

where $A_n=\sqrt{4\pi}A_{00n}$, and $a_{0n}=n\pi$, so that

$$t_{00n}=r_0^2/\left(n\pi\alpha\right)^2. \tag{10.71}$$

The initial condition is given by

$$\Delta T=T_0-T_f=\sum_{n=1}^{\infty}A_n j_l\left(a_{0n}r/r_0\right). \tag{10.72}$$

Therefore, the solution is given by

$$\begin{aligned} A_n &= \frac{2}{j_1^2(a_{0n})}\int_0^1 x^2 dx j_0\left(a_{0n} r / r_0\right)\Delta T \\ &= \frac{2}{a_{01} j_1(a_{0n})}\left(T_0 - T_f\right) \end{aligned} \tag{10.73}$$

or

$$T\left(\mathbf{r},t\right) = T_f + \sum_{n=1}^{\infty}\frac{2\Delta T}{a_{0n} j_1\left(a_{0n}\right)} j_0\left(a_{0n} r / r_0\right) e^{-\left(n\pi\alpha / r_0\right)^2 (t-t_0)}. \tag{10.74}$$

Figure 10-3 shows how the shape of the temperature distribution evolves in time. Initially con has a uniform temperature distribution, but the short decay time components quickly decay, leaving a slowly decaying component with roughly the shape of a $j_0(a_{01} r / r_0)$ Bessel function having a single maximum at the center of the sphere.

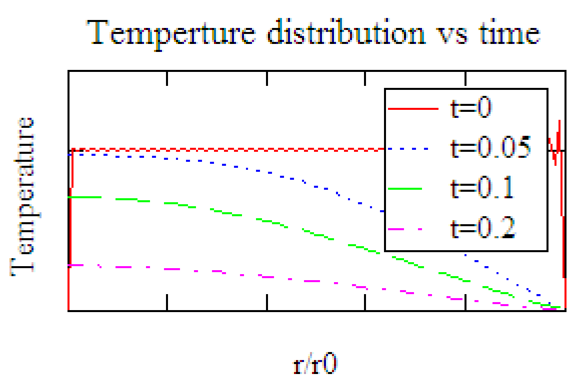


Figure 10-3 Temperature distribution in a sphere

10.6 Examples with cylindrical boundary conditions

❖ Normal modes in a cylindrical cavity

The normal frequencies of oscillation in a cylindrical cavity differ depending on whether the time-dependent equation satisfies Direchlet or Neumann boundary conditions. In either case, one is dealing with the interior solutions to the Helmholtz equation (10.8), therefore the solutions can be written in the general form

$$\Phi_k(\mathbf{r}) = J_m(k_{mn} r) e^{im\phi} \begin{Bmatrix} \cos n\pi z / L \\ \sin n\pi z / L \end{Bmatrix}. \tag{10.75}$$

For Direchlet Boundary conditions, the normal modes satisfy

$$\begin{aligned} &\Phi_k(\mathbf{r}) = J_m(k_{mn} r) e^{im\phi} \sin n\pi z / L \\ &\text{and } J_m(k_{mn} r_0) = 0, \end{aligned} \tag{10.76}$$

while, in the Neumann case, one has

$$\begin{aligned} &\Phi_k(\mathbf{r}) = J_m(k_{mn} r) e^{im\phi} \cos n\pi z / L \\ &\text{and } J'_m(k_{mn} r_0) = 0. \end{aligned} \tag{10.77}$$

❖ Temperature distribution in a cylinder

For time-independent cylindrical boundary conditions, the steady-state temperature $T_{ss}(\mathbf{r})$ is calculated as a solution to Laplace's equation, and the result subtracted from the initial temperature distribution within the cylindrical volume. The time-dependent temperature profile for a cylinder of radius r_0 and height L is then given by

$$\begin{aligned} &T(\mathbf{r},t)-T_{ss}(\mathbf{r}) \\ &=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}A_{mnl}J_m\left(a_{mn}r/r_0\right)e^{im\phi}\sin\frac{l\pi z}{L}e^{-t/t_{lmn}}, \end{aligned} \tag{10.78}$$

where a_{mn} are the zeroes of the Bessel functions and

$$\frac{1}{t_{mnl}}=\alpha^2\left(\left(\frac{a_{mn}}{r_0}\right)^2+\left(\frac{l\pi}{L}\right)^2\right)=\alpha^2k_{mnl}^2, \tag{10.79}$$

Orthogonality can be used to determine the coefficients of the time dependent part of the problem:

$$\begin{aligned} A_{mnl} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-im\phi} \\ &\times\frac{2}{J_{m-1}^2\left(a_{mn}\right)}\int_0^1 xdxJ_m\left(a_{mn}r/r_0\right) \\ &\times\frac{2}{L}\int_0^L dz\sin\frac{l\pi z}{L}\left[T(\mathbf{r},0)-T_{ss}(\mathbf{r})\right]. \end{aligned} \tag{10.80}$$

11. Green’s functions and propagators

When one exerts a force on a dynamic system, the response is a disturbance of the system that propagates in time. Up to now we have concentrated on the solution of linear homogeneous systems. But such systems do not start moving on their own. Homogeneous equations have the trivial solution that the field and its derivates vanish everywhere. Their motion arises from flow of energy and momentum into or out of the system, expressed in terms of boundary conditions, and ultimately, to the action of sources that are often inhomogeneous in origin. A complicated force acting for an extended period of time, or over an extended volume of space, can be decomposed into point-like impulses. If the equation is linear, the net effect can be expressed as a superposition of these influences. This is the essence of the Green’s function technique for solving inhomogeneous differential equations. A Green’s function represents the potential due to a point-like source meeting certain particular boundary conditions. If the equation is time dependent, the Green’s function is often referred to as a propagator. The positive time propagator propagates a signal into future times, and the negative time propagator propagates a signal backwards in time.

11.1 The driven oscillator

Consider a driven oscillator that might, for example, be an approximation to a swing with a child on it. When one pushes the swing, it begins to move. If one pushes in phase with a existing motion, the amplitude grows. Before and after the introduction of the time dependent force, assuming that the amplitude remains small, the motion of the swing is a solution to a linear homogeneous differential equation with a characteristic angular frequency of oscillation ω_o. It behaves like a driven oscillator. The differential equation of motion for the driven oscillator can be written as

$$\left(\frac{d^2}{dt^2}+\omega_0^2\right)y(t)=\begin{cases} f(t) & t>0, \\ 0 & t\le 0, \end{cases} \tag{11.1}$$

where $f(t)$ is a generalized force that begins acting at some time $t>0$. The initial state of the system is a solution to the homogeneous equation

$$\left(\frac{d^2}{dt^2}+\omega_0^2\right)y_h(t)=0, \tag{11.2}$$

with a solution

$$y_h(t)=A\cos\omega_0 t+B\sin\omega_0 t, \tag{11.3}$$

where the coefficients A and B can be determined from the initial conditions

$$y_h(0)=A, \quad y_h'(0)=\omega_0 B. \tag{11.4}$$

The complete solution to the inhomogeneous problem is a supposition of this homogeneous solution with a particular solution to the inhomogeneous problem that has the swing initially at rest.

$$y(t) = y_h(t) + y_p(t), \tag{11.5}$$

where

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right) y_p(t) = \begin{cases} f(t) & t > 0 \\ 0 & t \le 0 \end{cases} \tag{11.6}$$

and

$$y_p(t) = y_p'(t) = 0 \text{ for } t \le 0 \tag{11.7}$$

The solution to the driven oscillator problem can be expressed as a convolution over a simpler problem involving the response of the system to an impulsive force of unit magnitude acting at an instance of time $t' > 0$:

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right) g_+(t,t') = \delta(t - t'), \quad t' > 0 \tag{11.8}$$

satisfying the boundary condition

$$g_+(t,t') = 0 \quad t < t' \tag{11.9}$$

$g_+(t,t')$ is the positive time propagator that will propagate the solution forward in time. The general solution to the problem can then be written as

$$y(t) = y_h(t) + \int_0^t f(t') g(t,t') dt'. \tag{11.10}$$

The proof is straightforward:

$$\begin{aligned}\left(\frac{d^2}{dt^2}+\omega_0^2\right)y(t) &= \left(\frac{d^2}{dt^2}+\omega_0^2\right)y_h(t) \\ &\quad +\left(\frac{d^2}{dt^2}+\omega_0^2\right)\int_0^t f(t')g(t,t')dt' \\ &= 0+\int_0^t f(t')\left(\frac{d^2}{dt^2}+\omega_0^2\right)g(t,t')dt' \\ &= \int_0^t f(t')\delta(t-t')dt' = f(t).\end{aligned} \tag{11.11}$$

In another way of looking at the problem, the Green's function is a solution to the homogeneous equation for $t \neq t'$. Because of the delta function source term, it has a discontinuity in its derivative at $t = t'$:

$$\lim_{\varepsilon\to 0} y'(t)\Big|_{t'-\varepsilon}^{t'+\varepsilon} = \lim_{\varepsilon\to 0}\int_{t'-\varepsilon}^{t'+\varepsilon}\delta(t-t')dt' = 1. \tag{11.12}$$

Therefore, the solution can be written as

$$g_+(t,t') = \begin{cases} 0 & t < t' \\ \dfrac{\sin\omega_0 t}{\omega_0} & t > t', \end{cases} \tag{11.13}$$

or more compactly as

$$g_+(t,t') = \frac{\sin\omega_0 t}{\omega_0}\Theta(t-t') \tag{11.14}$$

where $\Theta(t-t')$ is the step function distribution given by

$$\Theta(t-t') = \begin{cases} 0 & t < t', \\ 1 & t > t'. \end{cases} \tag{11.15}$$

The step function satisfies the differential equation

$$\frac{d}{dt}\Theta(t-t') = \delta(t-t'), \tag{11.16}$$

which can be demonstrated by direct integration of the equation.

Suppose the swing were initially at rest, and that the force acts for a finite time $0 < t' < t_{\text{max}}$. The asymptotic state of the system can then be written as

$$\begin{aligned} y(t) &= \int_0^{t_{\text{max}}} f(t')g(t,t')dt' = \int_{t'>0}^{t_{\text{max}}} f(t')\frac{\sin\omega_0(t-t')}{\omega_0}dt' \\ &= y_0 \sin\left(\omega_0 t + \phi_0\right) \quad \text{for } t > t_{\text{max}}, \end{aligned} \tag{11.17}$$

where the solution for large times is a solution for the homogeneous equation with an amplitude and phase determined by the convolution of the green's function with the time dependent force over the period for which it was active.

It is unrealistic to expect a swing to oscillate forever, so let's introduce a subcritical damping force with a damping coefficient γ. The modified equation of motion is

$$\left(\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + \omega_0^2\right)y(t) = \begin{cases} f(t) & t > 0 \\ 0 & t \le 0, \end{cases} \tag{11.18}$$

which has the homogeneous solution

$$y_h(t) = Ae^{-\gamma t/2}\sin\left(\omega' t + \phi\right), \tag{11.19}$$

where

$$\omega' = \sqrt{\omega_0^2 + \frac{\gamma^2}{4}}. \tag{11.20}$$

The Green's function solution to the equation of motion is given by

$$g_+(t,t') = e^{-\gamma(t-t')/2} \frac{\sin \omega'(t-t')}{\omega'} \Theta(t-t'). \tag{11.21}$$

It is straight forward to show that

$$g_+(t,t') = 0 \quad t < t' \tag{11.22}$$

and

$$\lim_{\varepsilon \to 0} g'_+(t,t')\Big|_{t=t'-\varepsilon}^{t=t'+\varepsilon} = 1 \quad . \tag{11.23}$$

11.2 Frequency domain analysis

Another approach to this problem is to resolve the time spectra of the force into its frequency components. This leads to a Fourier transformation. Given an equation of the form

$$\left(\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + \omega_0^2\right) y(t) = f(t) \tag{11.24}$$

one can resolve the force into frequency components

$$f(t) = \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega. \tag{11.25}$$

Similarly, the response can be written as

$$y(t) = \int_{-\infty}^{\infty} y(\omega) e^{-i\omega t} d\omega. \tag{11.26}$$

Leading to the Fourier transform equation of motion

$$\left(-\omega^2 - i\gamma\omega + \omega_0^2\right) y(w) = f(\omega), \tag{11.27}$$

which has the solution

$$y(\omega) = \frac{f(\omega)}{\left(-\omega^2 - i\gamma\omega + \omega_0^2\right)} = \Gamma_R(\omega, \omega_0) f(\omega). \tag{11.28}$$

The response at a given frequency has a typical resonance line shape, as seen in Figure 11-1, where the norm-square of Γ_R.is plotted By making the inverse transform, one gets the particular solution

$$y_p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} y(\omega) e^{i\omega t} d\omega. \tag{11.29}$$

The boundary conditions can be satisfied by adding an appropriate homogeneous term to this solution.

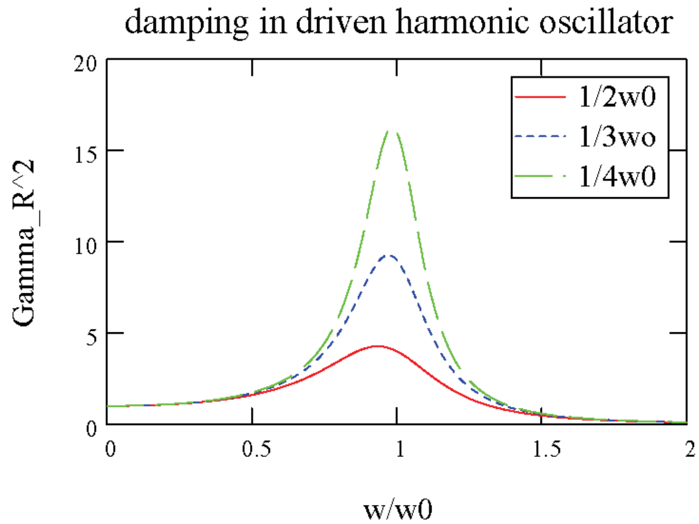


Figure 11-1 Resonance response of a driven oscillator for different damping constants.

11.3 Green's function solution to Possion's equation

Gauss's Law for the divergence of the electric field in the presence of a charge distribution can be expressed by Poisson's equation

$$\nabla \bullet \mathbf{E} = -\nabla^2 \Phi(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0}. \tag{11.30}$$

The electrostatic potential $\Phi(\mathbf{r})$ of a point charge of magnitude q and position $\mathbf{r}'$ in free space is given by

$$\Phi(\mathbf{r},\mathbf{r}') = \frac{q}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}. \tag{11.31}$$

The potential due to a distribution of charge with density $\rho(\mathbf{r}')$ can be written as a integral over the pointlike potential contributions for infinitesimal elements of charge $dq = \rho(\mathbf{r}')\,d^3r'$, giving

$$\Phi(\mathbf{r}) = \int d^3r' \frac{\rho(\mathbf{r}')}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}. \tag{11.32}$$

From this we deduce that the free space Green's function for Poisson's equation is given by

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}, \tag{11.33}$$

where

$$-\nabla^2 G(\mathbf{r},\mathbf{r}') = \frac{1}{\varepsilon_0}\delta(\mathbf{r}-\mathbf{r}') \tag{11.34}$$

and

$$\Phi(\mathbf{r}) = \int d^3r' G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}'). \tag{11.35}$$

11.4 Multipole expansion of a charge distribution

Using the series expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{l,m}^*(\theta',\phi') Y_{l,m}(\theta,\phi), \tag{11.36}$$

one can make a multipole expansion of an arbitrary charge distribution, assuming that the charge distribution is localized within a volume of radius r_o. We are interested in finding the potential only in the exterior region $r > r_o > r'$. Then equation (11.36) can be written as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r'^l}{r^{l+1}}Y_{l,m}^*(\theta',\phi')Y_{l,m}(\theta,\phi). \qquad (11.37)$$

Substituting into equation (11.32) gives

$$\begin{aligned}\Phi(\mathbf{r}) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{Y_{l,m}(\theta,\phi)}{r^{l+1}}\int d^3r' r'^l \frac{\rho(\mathbf{r}')}{\varepsilon_0}Y_{l,m}^*(\theta',\phi') \\ &= \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}B_{lm}\left(\frac{1}{r}\right)^{l+1}Y_{l,m}(\theta,\phi), \text{ implying} \qquad (11.38) \\ B_{lm} &= \frac{4\pi}{2l+1}\int d^3r'^l\rho(\mathbf{r}')Y_{l,m}^*(\theta',\phi'),\end{aligned}$$

where the B_{lm} represent the multipole moments of the distribution.

As an example, consider the following line charge distribution along the z-axis

$$\rho(\mathbf{r}') = qz/a^2\delta(x)\delta(y) \text{ for } |\mathrm{z}| < a. \qquad (11.39)$$

We are interested in obtaining the multipole expansion of this distribution for $|\mathbf{r}| > a$. By azimulthal symmetry, only the $m = 0$ terms will contribute.

$$\begin{aligned}
B_{l0} &= \frac{4\pi}{2l+1}\int d^3r' r'^l \rho(\mathbf{r}') Y^*_{l,0}(\theta',\phi') \\
&= \frac{4\pi}{2l+1}\int_{-a}^{a} dz' \iint dx' dy' r'^l \left(\frac{qz'}{a^2}\right)\delta(x')\delta(y') Y^*_{l,0}(\theta',\phi') \\
&= \frac{4\pi q/a^2}{2l+1}\int_{-a}^{a} dz' z'^{l+1}\sqrt{\frac{2l+1}{4\pi}} P_l(z/|z|) \\
&= \sqrt{\frac{4\pi}{2l+1}}\frac{q}{a^2}\left[\int_0^a dz' z'^{l+1} - \int_{-a}^{0} dz' z'^{l+1}\right],
\end{aligned} \tag{11.40}$$

or

$$B_{l0} = \begin{cases} 0 & \text{for even } l, \\ \sqrt{\dfrac{4\pi}{2l+1}}\dfrac{2qa^l}{(l+2)} & \text{for odd } l, \end{cases} \tag{11.41}$$

Therefore,

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 a}\sum_{\text{even } l}^{\infty}\frac{2}{l+2}P_l(\cos\theta)\left(\frac{a}{r}\right)^{l+1} \quad \text{for } |\mathbf{r}| > a. \tag{11.42}$$

For large r, the leading order behavior of the distribution approaches that of a dipole charge distribution

$$\Phi(\mathbf{r}) = \frac{qa\cos\theta}{6\pi\varepsilon_0 r^2}. \tag{11.43}$$

11.5 Method of images

The Free space Green's function is a solution to Poisson's equation for a unit point charge, subject to the boundary conditions

$$\lim_{r\to\infty} G_{Free}(\mathbf{r},\mathbf{r}') = 0. \tag{11.44}$$

To find a similar Green's function for a unit point charge within a closed surface, subject to Direchlet Boundary conditions at the surface, this Green's function must be modified to vanish at the boundary. This can be accomplished by adding a solution of the homogeneous equation, valid within the boundary, to the free space Green's function:

$$G_{Direchlet}\left(\mathbf{r},\mathbf{r}'\right)=G_{Free}\left(\mathbf{r},\mathbf{r}'\right)+\Phi_h\left(\mathbf{r}\right), \tag{11.45}$$

Subject to the constraint

$$G_{Direchlet}\left(\mathbf{r},\mathbf{r}'\right)\Big|_{Boundary}=0. \tag{11.46}$$

The general solution to Poisson's equation within the boundary region is given by

$$\Phi\left(\mathbf{r}\right)=V_{\mathrm{h}}\left(\mathbf{r}\right)+\int d^3r' G_{Direchlet}\left(\mathbf{r},\mathbf{r}'\right)\rho\left(\mathbf{r}'\right). \tag{11.47}$$

Where $V_{\mathrm{h}}\left(\mathbf{r}\right)$ is another solution to the homogeneous Laplace equation satisfying the actual Direchlet boundary on the boundary surface:

$$\Phi\left(\mathbf{r}\right)\Big|_{boundary}=V_{\mathrm{h}}\left(\mathbf{r}\right)\Big|_{boundary} \tag{11.48}$$

❖ Solution for a infinite grounded plane

Calculating Green's functions of a complicated surface is non trivial, but for simple surface, one can use symmetry arguments to generate an appropriate Green's function. For example, suppose the boundary is a grounded infinite plane at $z = 0$, and we were interested in obtaining the Green's function for the positive half plane $z \geq 0$. The surface of the plane is an equipotential surface, therefore the Electric field would have to be normal to the surface (if the field has a component in the plane, charge would flow, which would contradict the assumption that the system has reached static equilibrium).

The grounded plane problem for the positive half plane would be equivalent to removing the plane and adding a mirror charge of opposite sign in the negative half plane. In fact for any distribution of charge $\rho(x, y, z)$ in the positive half plane, the mirror distribution $-\rho(x, y, -z)$ would lead to a zero-valued, equipotential surface at $z = 0$. In the case of a point charge at $\mathbf{r}' = (x', y', z')$, where $z' \geq 0$, one can place an image charge of opposite sign at $\mathbf{r}'' = (x', y', -z')$ to construct the Green's function

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}') &= \frac{1}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|} - \frac{1}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}''|} \\ &= \frac{1}{4\pi\varepsilon_0 \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \\ &\quad - \frac{1}{4\pi\varepsilon_0 \sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}}. \end{aligned} \tag{11.49}$$

Note

$$-\nabla^2 G(\mathbf{r}, \mathbf{r}') = \frac{1}{\varepsilon_0} \delta(\mathbf{r} - \mathbf{r}') \text{ for } z > 0 \tag{11.50}$$

and

$$G(\mathbf{r},\mathbf{r}')\Big|_{z=0} = 0. \tag{11.51}$$

❖ Induced charge distribution on a grounded plane

The induced charge density on the conducting plane is given by

$$E_z\Big|_{z=0} = -\frac{\partial\Phi(\mathbf{r})}{\partial z}\Bigg|_{z=0} = \frac{\sigma(x,y)}{\varepsilon_0} \tag{11.52}$$

Therefore, a point particle of magnitude q located at $\mathbf{r}'$ induces a surface charge density given by

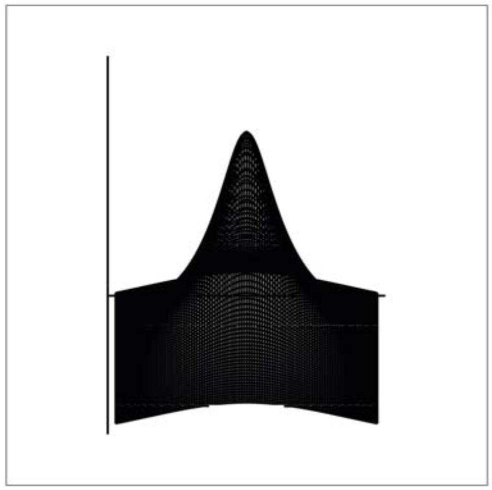
$$\sigma(x,y) = -q\varepsilon_0 \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial z}\Bigg|_{z=0}, \tag{11.53}$$

$$\sigma(x,y) = \frac{-2qz'}{4\pi\left((x-x')^2+(y-y')^2+(z')^2\right)^{3/2}}. \tag{11.54}$$

Integrating the induced charge density over the surface gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\sigma(x,y) = \int_0^{2\pi}\int_0^{\infty} \rho d\rho d\phi \frac{-2qz'}{4\pi\left(\rho^2+z'^2\right)^{3/2}} = -q. \tag{11.55}$$

A point charge induces a net charge of equal magnitude and opposite sign on the conducting surface. This is illustrated in Figure 11-1,which shows how a positive charge attracts a negative charge density of equivalent magnitude to the surface region closest to it. The sharpness of the induced charge distribution depends on how close the point charge is to the plane.



11-2 Induced surface charge density on a grounded plane due to a nearby point charge.

❖ Green's function for a conducting sphere

The above technique is called the method of images. It can be extended to find the Green's function for a grounded spherical cavity. Let the radius of the sphere be a and let $\mathbf{r}'$ be the position of a point charge inside the cavity. Then one can construct an image charge of magnitude q'' and position $\mathbf{r}''=\lambda\mathbf{r}'$ where λ is some scale factor to give the Green's function solution

$$G_{sphere}\left(\mathbf{r},\mathbf{r}'\right)=\frac{1}{4\pi\varepsilon_0\left|\mathbf{r}-\mathbf{r}'\right|}-\frac{q''}{4\pi\varepsilon_0\left|\mathbf{r}-\mathbf{r}''\right|}, \qquad (11.56)$$

subject to the constraint

$$G_{sphere}\left(\mathbf{r},\mathbf{r}'\right)\Big|_{r=a}=0. \tag{11.57}$$

Letting $x=\cos\theta$, we can rewrite the potential in terms of the generating function for the Legendre polynomials:

$$G_{sphere}\left(\mathbf{r},\mathbf{r}'\right)=\frac{1}{4\pi\varepsilon_0 r\left|1-2xh'+h'^2\right|^{1/2}}-\frac{q''}{4\pi\varepsilon_0 r\left|1-2xh''+h''^2\right|^{1/2}}, \tag{11.58}$$

where $h'=r'/r$ and $h''=r'/r$. Using the geometric ratio $r'r''=a^2$, so that $\lambda=\left(a/r'\right)^2$, or

$$\mathbf{r}''=\mathbf{r}'\frac{a^2}{r'^2}, \tag{11.59}$$

gives

$$G_{sphere}\left(\mathbf{r},\mathbf{r}'\right)\Big|_{r=a}=\frac{1}{4\pi\varepsilon_0 a}\left(\frac{1}{\left|1-2x\frac{r'}{a}+\left(\frac{r'}{a}\right)^2\right|^{1/2}}-\frac{q''}{\left|1-2x\frac{a}{r'}+\left(\frac{a}{r'}\right)^2\right|^{1/2}}\right), \tag{11.60}$$

which reduces to

$$G_{sphere}\left(\mathbf{r},\mathbf{r}'\right)\Big|_{r=a}=\frac{1}{4\pi\varepsilon_0 a}\frac{1}{\left|1-2x\frac{r'}{a}+\left(\frac{r'}{a}\right)^2\right|^{1/2}}\left(1-\left(\frac{r'}{a}\right)q''\right)=0, \tag{11.61}$$

The latter condition is satisfied when

$$q''=\left(\frac{a}{r'}\right). \tag{11.62}$$

Therefore,

$$G_{sphere}(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|} - \frac{\left(\frac{a}{r'}\right)}{4\pi\varepsilon_0 \left|\mathbf{r}-\left(\frac{a}{r'}\right)^2 \mathbf{r}'\right|}. \tag{11.63}$$

11.6 Green's function solution to the Yakawa interaction

The strong nuclear force, unlike the electromagnetic force, is short ranged. This short range character is due to a massive boson interaction. To model this, in the static limit, we add a mass term to the Laplace equation, giving rise to a Yukawa interaction,

$$-\left(\nabla^2 - m^2\right)\Phi(\mathbf{r}) = \rho(\mathbf{r}). \tag{11.64}$$

The factor m results in a exponential damping of the potential, giving it a short range character. This becomes apparent when one solves for the Green's function

$$-\left(\nabla^2 - m^2\right)G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}'), \tag{11.65}$$

which results in the free space Green's function

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{-m|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{11.66}$$

Letting $m \to 0$ recovers the Coulomb result in units of $\varepsilon_0 = 1$. Therefore, the long range character of the electromagnetic force is due to the photon being massless.